Journal of Engg. Research Vol. 9 No. (3A) September 2021 pp. 52-63 DOI : 10.36909/jer.v9i3A.8201

Dynamical analysis of the general beam model with singularity function

Duygu DÖNMEZ DEMİR*,*, B. Gültekin SINIR** and Emine KAHRAMAN*

**Department of Mathematics, Manisa Celal Bayar University, Manisa, Turkey*
***Department of Civil Engineering, Manisa Celal Bayar University, Manisa, Turkey*
**Corresponding Author: duygu.donmez@cbu.edu.tr*

Submitted: 02/05/2019
Revised: 23/12/2020
Accepted: 04/01/2021

ABSTRACT

The aim of this study is to present a general model with variable coefficients corresponding to some structural elements such as beam, string, bar, and rod. To solve general model with variable coefficients, a different solution procedure combining a method of multiple scales (MMS) and a finite difference method (FDM) is presented in this study. This technique provides an advantage in the numerical solution of the structural element model containing any discontinuity and in its dynamical analysis by perturbation method. Furthermore, two problems including discontinuity are considered to show the accuracy of the method presented. The comparisons of the numerical results obtained from the proposed method and classical method are introduced.

Keywords: General model; Method of multiple scales; Finite differences method; Singularity function; Discontinuous beams.

INTRODUCTION

Many engineering structures consist of the structural elements involving some discontinuities. The dynamic modelling of these structures leads to a set of partial differential equations (PDE). Generally, the mathematical models of this type do not contain the discontinuous functions such as Dirac delta and Heaviside step function; therefore, it can be solved analytically in a linear case. The same structures may be modeled by single differential equation instead of a set of equations when the system has discontinuities. In this case, a numerical method is needed in analyzing the model including singularity function.

Recently, Dinev (2012) has suggested an approach for analytical solution of the problem of bending a beam on an elastic foundation using singularity functions. Caddemi and Morassi (2013) proposed a verification of the rotational elastic spring model of an open crack formulated by suitable Dirac's Delta functions in a beam in bending deformation. Caddemi et al. (2013a) presented the closed-form solutions of the Timoshenko beam model subject to internal singularities leading to deflection and rotation discontinuities. Besides, a model of the stepped Timoshenko beam under deflection and rotation discontinuities that adopted Heaviside's and Dirac's Delta functions along the span was suggested by Caddemi et al. (2013b). For static and vibration analyses of stepped beam using singularity functions, Cheng et al. (2014) introduced a systematic approach, which is performed by directly differentiating the analytical deflection function with respect to any beam-related design variable.

In this study, a general model corresponding to the structural elements with different discontinuities is introduced. The general solution procedure is more effective instead of solving each problem, separately. For this purpose, many researchers (Lacarbonara, 1999, Hosseini & Zamanian, 2013, Ghayesh et al., 2012) have introduced some general models with operator notations, which are suitable for perturbation calculations. A different approach is presented to solve the general model proposed in this study. This approach has the advantages of the numerical method in the discontinuity case, as well as the perturbation technique in the dynamical analysis. The mathematical model of the problem containing discontinuity can be derived in two different ways. First, the system is separated from each point including discontinuity, and the equation is separately written for each part. Thus, the number $M + 1$ of equation is written, where the number of discontinuities equals M. The total number of boundary and transient conditions is $4\times(M + 1)$. Applying these conditions, the system of the linear algebraic equations is obtained as four times of span numbers. It is almost impossible to solve such a system for the large numbers M. Second, instead of a set of equations, the problem can be modeled by one equation including singularity function. Firstly, the MMS is directly applied to the equation having singularity function. At the first order, a linear equation is obtained. Substituting the assumption of first order into the equation yields an ordinary differential equation having singularity function. Since the solution of the differential equation with the singularity function is difficult to obtain analytically, numerical methods are needed. For this purpose, the finite difference method is used in this study.

GOVERNING EQUATION

The general structural dynamic model (Sınır, 2015) including harmonically internal and external excitations is

$$a(x)\ddot{y}+L_1[y]+\varepsilon\{L_2[y]\cos\Omega_1 t+F(x)\cos\Omega_2 t+L_3[\dot{y}]\}=0 \tag{1}$$

$$B_{11}[y]=B_{12}[y]=0 \text{ and } B_{21}[y]=B_{22}[y]=0 \tag{2}$$

where $y(x,t)$ is the transverse deflection, x is an axis, which denotes direction of the structure element, and t is the time variable, and $F(x)$ is the amplitude of external force. $L_2[y]\cos\Omega_1 t$ and $F(x)\cos\Omega_2 t$ correspond to parametric excitation and harmonically external force, respectively. Ω_1 and Ω_2 are the frequencies of the internal and external excitation, respectively. ε is a small dimensionless parameter. The dot represents differentiation with respect to time t. The subscripts of i, j at boundary conditions represent jth condition of the ith support for B_{ij}. $a(x)$ is the arbitrary function representing the variation of the mass and cross section. L_1 is related to stiffness structural elements. L_3 represents viscoelastic properties of the system. $a(x)$ and the operator L_1 may contain the singularity function to define discontinuities of one-dimensional structure (such as crack, stepped beam, or multisupport). L_1, L_2 and L_3 are linear and self-adjoint differential operators having variable coefficients. The differential operator L having variable coefficients can be demonstrated as

$$L=p_4(x)\frac{d^4}{dx^4}+p_3(x)\frac{d^3}{dx^3}+p_2(x)\frac{d^2}{dx^2}+p_1(x)\frac{d}{dx}+p_0(x)$$

where the highest derivative is denoted by fourth order since the mathematical model of the beams as structural elements is considered (L_1, L_2 and L_3 can be in form of the operator L). The space domain is considered as [0,1]. Also, these linear operators correspond to the other one-dimensional structural element such as bar, string, and cable. For example, p_3 and p_4 vanish for a bar or string structure element.

METHOD OF MULTIPLE SCALES

In this section, MMS is directly applied to the governing equation. Perturbation series expansion is assumed as

$$y(x,T_0,T_1;\varepsilon)=y_0(x,T_0,T_1;\varepsilon)+\varepsilon\, y_1(x,T_0,T_1;\varepsilon)+\ldots \tag{3}$$

where T_n is different time scales in form of $T_n=\varepsilon^n t$. The time derivatives in terms of the new time scales are defined by

$$\frac{d}{dt} = D_0 + \varepsilon D_1 + \cdots \text{ and } \frac{d^2}{dt^2} = D_0^2 + 2\varepsilon D_0 D_1 + \cdots \tag{4}$$

where $D_i = \partial / \partial T_i$. Substituting Eqs. (3-4) into Eq. (1-2) and separating terms at each order ε yield

$$O(1):\ a(x) D_0^2 y_0 + L_1[y_0] = 0 \tag{5}$$

$$B_{11}[y_0] = B_{12}[y_0] = 0 \text{ and } B_{21}[y_0] = B_{22}[y_0] = 0 \tag{6}$$

$$O(\varepsilon):\ a(x) D_0^2 y_1 + L_1[y_1] = -L_3[D_0 y_0] - 2a(x) D_0 D_1 y_0 - L_2[y_0]\cos\Omega_1 T_0 - F(x)\cos\Omega_2 T_0 \tag{7}$$

$$B_{11}[y_1] = B_{12}[y_1] = 0 \text{ and } B_{21}[y_1] = B_{22}[y_1] = 0 \tag{8}$$

The solution at first order is assumed as

$$y_0(x, \mathrm{T}_0, \mathrm{T}_1) = \left(A_n(\mathrm{T}_1) e^{i\omega_n \mathrm{T}_0} + \bar{A}_n(\mathrm{T}_1) e^{-i\omega_n \mathrm{T}_0}\right) X_n(x);\ n = 1,2,3,\ldots \tag{9}$$

where A_n is the complex amplitude. Substituting Eq. (9) into Eq. (5), the relation in the following is obtained.

$$L_1[X_n] - a(x)\omega_n^2 X_n = 0 \tag{10}$$

Eq. (10) can be called as an eigenvalue-eigenfunction problem. ω_n represents the eigenvalues, and X_n corresponds to eigenfunctions of the system. From Eqs. (10), X_n and ω_n can be calculated. Since the resulting equation is the ordinary differential equation with variable coefficient, finite differences method is used for the approximate solution.

FINITE DIFFERENCES METHOD

For finite differences method, there are three different finite differences schemes: forward differences, backward differences, and central differences. For small truncation error, the central difference is chosen. Then, first four derivatives are given as follows:

$$X'_j \approx \frac{X_{j+1} - X_{j-1}}{2\Delta x} \text{ and } X''_j \approx \frac{X_{j+1} - 2X_j + X_{j-1}}{\Delta x^2} \tag{11.a}$$

$$X'''_j = \frac{X_{j+2} - 2X_{j+1} + 2X_{j-1} - X_{j-2}}{2\Delta x^3} \text{ and } X^{iv}_j = \frac{X_{j+2} - 4X_{j+1} + 6X_j - 4X_{j-1} + X_{j-2}}{\Delta x^4} \tag{11.b}$$

where $\Delta x = 1/N$. N is the total number of short segments into system. In these discretized forms, the subscript indicates spatial node. Substituting Eq. (11) into Eq. (10) yields the discretized equation at jth spatial node as

$$b_{4,j} X_{n,j+2} + b_{3,j} X_{n,j+1} + b_{2,j} X_{n,j} + b_{1,j} X_{n,j-1} + b_{0,j} X_{n,j-2} = 0 \tag{12}$$

where

$$b_{0,j} = \frac{p_{4,j}}{\Delta x^4} - \frac{p_{3,j}}{2\Delta x^3},\ b_{1,j} = -\frac{4p_{4,j}}{\Delta x^4} + \frac{p_{3,j}}{\Delta x^3} + \frac{p_{2,j}}{\Delta x^2} - \frac{p_{1,j}}{2\Delta x},\ b_{2,j} = \frac{6p_{4,j}}{\Delta x^4} - \frac{2p_{2,j}}{\Delta x^2} + p_{0,j} \tag{13.a}$$

$$b_{3,j} = -\frac{4p_{4,j}}{\Delta x^4} - \frac{p_{3,j}}{\Delta x^3} + \frac{p_{2,j}}{\Delta x^2} + \frac{p_{1,j}}{2\Delta x},\ b_{4,j} = \frac{p_{4,j}}{\Delta x^4} + \frac{p_{3,j}}{2\Delta x^3} \tag{13.b}$$

The boundary conditions are

$$B_{11}(X_n)=X_n,\ B_{12}(X_n)=X''_n,\ B_{21}(X_n)=X_n,\ B_{22}(X_n)=X''_n, \tag{14}$$

$$B_{11}(X_n)=X_n,\ B_{12}(X_n)=X'_n,\ B_{21}(X_n)=X_n,\ B_{22}(X_n)=X'_n, \tag{15}$$

$$B_{11}(X_n)=X_n,\ B_{12}(X_n)=X''_n,\ B_{21}(X_n)=X_n,\ B_{22}(X_n)=X'_n \tag{16}$$

for pinned-pinned (PP) support, clamped-clamped (CC) support, and pinned-clamped (PC) support, respectively. Using boundary conditions, Eq. (12) yields the algebraic equation system. In this study, the algebraic equations system in matrix form is given for PP, CC, and PC support conditions. Finite differences applications of the boundary conditions are as follows:

$$X_{n,0}=0,\ X_{n,-1}=-X_{n,1},\ X_{n,N}=0,\ X_{n,N+1}=-X_{n,N-1} \tag{17}$$

$$X_{n,0}=0,\ X_{n,-1}=X_{n,1},\ X_{n,N}=0,\ X_{n,N+1}=X_{n,N-1} \tag{18}$$

$$X_{n,0}=0,\ X_{n,-1}=-X_{n,1},\ X_{n,N}=0,\ X_{n,N+1}=X_{n,N-1} \tag{19}$$

for PP, CC, and PC support, respectively. Using the boundary conditions (17) and the relations (12-13), the algebraic equation system can be written in the following matrix form:

$$\begin{bmatrix} b_{2,1}-b_{0,1} & b_{3,1} & b_{4,1} & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ b_{1,2} & b_{2,2} & b_{3,2} & b_{4,2} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ b_{0,3} & b_{1,3} & b_{2,3} & b_{3,3} & b_{4,3} & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & b_{0,4} & b_{1,4} & b_{2,4} & b_{3,4} & b_{4,4} & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & b_{0,N-4} & b_{1,N-4} & b_{2,N-4} & b_{3,N-4} & b_{4,N-4} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & b_{0,N-3} & b_{1,N-3} & b_{2,N-3} & b_{3,N-3} & b_{4,N-3} \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & b_{0,N-2} & b_{1,N-2} & b_{2,N-2} & b_{3,N-2} \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & b_{0,N-1} & b_{1,N-1} & b_{2,N-1}-b_{4,N-1} \end{bmatrix} \begin{bmatrix} X_{n,1} \\ X_{n,2} \\ X_{n,3} \\ X_{n,4} \\ \vdots \\ X_{n,N-4} \\ X_{n,N-3} \\ X_{n,N-2} \\ X_{n,N-1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{20}$$

Similarly, the other algebraic equation systems for CC and PC can be obtained. The determinant of the matrices of the coefficients (20) must be equal to zero for nontrivial solutions. The mode shapes X_n are obtained depending on the node $N/2$. Thus, y_0 is determined from the solution of Eq. (5). For the solution of ε -order, substituting Eq. (9) into Eq. (7) yields

$$\begin{aligned} a(x)D_0^2 y_1 + L_1[y_1] = &-\left(2a(x)\,i\,\omega_n X_n D_1 A_n + i\,\omega_n A_n L_3[X_n]\right)e^{i\omega_n T_0} \\ &-\frac{L_2[X_n]}{2}\left[A_n\left(e^{i(\omega_n+\Omega_1)T_0}+e^{i(\omega_n-\Omega_1)T_0}\right)\right]-\frac{F(x)}{2}e^{i\Omega_2 T_0}+cc \end{aligned} \tag{21}$$

where cc denotes complex conjugates. The solution of Eq. (21) can be considered as

$$y_1(x,T_0,T_1)=\varphi_n(x,T_1)e^{i\omega_n T_0}+Y(x,T_0,T_1)+cc \tag{22}$$

where the first and second terms are related to secular and nonsecular terms, respectively. The term related to force is either secular or nonsecular term depending on its frequency.

CASE STUDIES

Applying solvability conditions (Nayfeh, 1981), some definite integrals including mode shapes X_n and its derivatives are obtained in the form of

$$\alpha_{in} = \int_0^1 X_n L_i \left[X_n \right] dx \tag{23}$$

where only numerical values of X_n exist. In the approximate calculation of integrals from the solvability condition, Simpson's method, one of the numerical integration rules, was used. The formula of Simpson's method is given as

$$\int_0^1 f(x)\,dx \cong \frac{\Delta x}{3}\left[f_0 + 4\left(f_1 + f_3 + \ldots + f_{2N-1}\right) + 2\left(f_2 + f_4 + \ldots + f_{2N}\right) + f_N \right] \tag{24}$$

where N is the number of subintervals, Δx is the step length ($\Delta x = 1 / N$), and the function f_j corresponds to

$$\begin{aligned} f_j = \frac{X_{n,j}}{\Delta x^4}\Big[& X_{n,j+2}\left(\Delta x\, p_3(x_j) + p_4(x_j)\right) + X_{n,j+1}\left(\Delta x^3 p_1(x_j) + \Delta x^2 p_2(x_j)\right. \\ & \left. -2\Delta x\, p_3(x_j) - 4P_4(x_j)\right) + X_{n,j}\left(\Delta x^4 p_0(x_j) - 2\Delta x^2 p_2(x_j) + 6 p_4(x_j)\right) \\ & + X_{n,j-1}\left(-\Delta x^3 p_1(x_j) + \Delta x^2 p_2(x_j) + 2\Delta x\, p_3(x_j) - 4 p_4(x_j)\right) \\ & + X_{n,j-2}\left(-\Delta x\, p_3(x_j) + p_4(x_j)\right)\Big] \end{aligned} \tag{25}$$

Ω_1 AWAY FROM 0 AND $2\omega_n$, Ω_2 AWAY FROM ω_n

Substituting Eq. (9) into Eq. (7) and eliminating the terms producing secularity yield the equation in the following:

$$L_1(\varphi_n) - a(x)\omega_n^2 \varphi_n = -2a(x) i\omega_n X_n D_1 A_n - i\omega_n L_3(X_n) A_n \tag{26}$$

The solvability condition requires (see reference (Nayfeh, 1981) for calculating in detailed)

$$D_1 A_n + \alpha_{3n} A_n = 0 \tag{27}$$

where the normalization is $\int_0^1 a(x) X_n^2 dx = 1$, and the coefficient α_{3n} is $\alpha_{3n} = \frac{1}{2}\int_0^1 X_n L_3 \left[X_n \right] dx$.

Then, A_n is obtained from Eq. (27) as follows:

$$A_n = C e^{-\alpha_{3n} T_1} \tag{28}$$

where C is an arbitrary constant, and α_{3n} is always real and positive. Then, the amplitude of the system exponentially decreases, and the solution is stable.

Ω_1 CLOSE TO 0, Ω_2 AWAY FROM ω_n

In this case, $\Omega_1 \approx 0$ and $\Omega_2 \neq \omega_n$ are assumed, where the detuning parameter σ is used to describe the nearness Ω_1 to 0. Then, $\Omega_1 = \varepsilon\sigma$. Similarly, the solvability condition is obtained as follows:

$$D_1 A_n + \left(\alpha_{3n} + 2i\alpha_{2n} \cos\sigma T_1\right) A_n = 0 \tag{29}$$

where $\alpha_{2n} = -\frac{1}{4\omega_n}\int_0^1 X_n L_2 \left[X_n \right] dx$. From the solution of Eq. (29), the amplitude is found as

$$A_n = A_0 e^{-\alpha_{3n}T_1 - \frac{2i\alpha_{2n}}{\sigma}\sin\sigma T_1}. \tag{30}$$

The term $A_0 e^{-\alpha_{3n}T_1}$ leads to diminish the amplitude. Since $|\sin\sigma T_1| \le 1$, the complex amplitudes are bounded in time. However, this order of approximation has not got any instability.

Ω_1 CLOSED TO $2\omega_n$, Ω_2 AWAY FROM ω_n

The excitation frequency has the variation around two times of the natural frequencies such that $\Omega_1 = 2\omega_n + \varepsilon\sigma$, where σ is the detuning parameter. Thus, one obtains

$$D_1 A_n + \alpha_{3n} A_n + i\alpha_{2n} e^{i\sigma T_1} \bar{A}_n = 0. \tag{31}$$

The polar form of A_n is assumed as

$$A_n = \frac{1}{2} a_n (T_1) e^{i\beta_n(T_1)} \tag{32}$$

Substituting Eq. (32) into Eq. (31) and separating real and imaginary parts of the resulting equation yield

$$\frac{da_n}{dT_1} + \alpha_{3n} a_n - a_n \alpha_{2nI} \sin\gamma = 0 \tag{33}$$

$$\sigma - \frac{d\gamma}{dT_1} + \alpha_{2nI} \cos\gamma = 0 \tag{34}$$

where $\gamma = \sigma T_1 - 2\beta$. Since da_n/dT_1 and $d\gamma/dT_1$ should be equal to zero for steady state solutions, the detuning parameter is obtained as

$$\sigma = \mp\sqrt{\alpha_{2nI}^2 - \alpha_{3n}^2}. \tag{35}$$

Ω_2 CLOSED TO ω_n, Ω_1 AWAY FROM $2\omega_n$ AND 0

For $\Omega_2 = \omega_n + \varepsilon\sigma$, the amplitude equation is obtained as follows:

$$D_1 A_n + \alpha_{3n} A_n = \frac{1}{2} i\alpha_{1n} e^{i\sigma T_1} \tag{36}$$

where $\alpha_{1n} = \frac{1}{2\omega_n}\int_0^1 F(x) X_n dx$. Substituting Eq. (32) into Eq. (36) and separating real and imaginary parts, the resulting equation is

$$\frac{da_n}{dT_1} + \alpha_{3n} a_n = -\alpha_{1nI} \sin\gamma \tag{37}$$

$$\sigma a_n - \frac{d\gamma}{dT_1} a_n = \alpha_{1nI} \cos\gamma \tag{38}$$

where $\gamma = \sigma T_1 - \beta$. Thus, σ is found as

$$\sigma = \mp\frac{1}{a_n}\sqrt{\alpha_{1nI}^2 - a_n^2 \alpha_{3n}^2} \tag{39}$$

considering da_n/dT_1 and $d\gamma/dT_1$ equal to zero for the steady state solutions.

BEAMS WITH MULTIPLE SPAN UNDER VERTICAL SPRING-SUPPORT SUBJECTED TO PARAMETRIC AXIAL FORCE

In this section, the beam having linear elastic spring at internal support (Sınır & Sınır, 2011) is considered. The equation of motion involving a singularity function for the number M of linear elastic spring is presented as follows:

$$EI\hat{y}^{iv}+\hat{P}\hat{y}''+\sum_{j=1}^{M}\hat{k}_j\,\delta\left(\hat{x}-\hat{\eta}_j\right)\hat{y}+m\ddot{\hat{y}}+\hat{\mu}\dot{\hat{y}}=0;\;\; j=1,2,\ldots,M \tag{40}$$

and boundary conditions

$$\hat{y}(0,\hat{t})=\hat{y}''(0,\hat{t})=0 \text{ and } \hat{y}(1,\hat{t})=\hat{y}''(1,\hat{t})=0 \tag{41}$$

where E and I are the modulus of elasticity and the moment of inertia, respectively. m denotes the mass per unit length. $\hat{\mu}$ is the linear viscous damping coefficient. L describes the distance apart between two simple supports. $\hat{P}(\hat{t})$ is axially harmonic compressed by a loading such that $\hat{P}=\hat{P}_0+\varepsilon\hat{P}_1\cos\Omega\hat{t}$, where ε is a small dimensionless parameter. $\hat{k}$ represents the spring constant. δ denotes Dirac delta function corresponding to singularity function $\left\langle \hat{x}-\hat{\eta}_j\right\rangle^{-1}$ (Lect. 12). The singularity functions are used to calculate deflections of beams with various loading and support conditions. Introducing the dimensionless terms for Eq. (40-41)

$$y=\frac{\hat{y}}{L},\ x=\frac{\hat{x}}{L},\ t=\frac{\hat{t}}{L^2}\sqrt{\frac{EI}{m}} \text{ and } \eta_j=\frac{\hat{\eta}_j}{L},\ k_j=\frac{\hat{k}_jL^3}{EI},\ P=\frac{\hat{P}L^2}{EI},\ \mu=\frac{\hat{\mu}L^2}{\sqrt{E\,Im}}, \tag{42}$$

the resulting equation is obtained as

$$y^{iv}+Py''+\sum_{j=1}^{M}k_j\delta\left(x-\eta_j\right)y+\ddot{y}+\mu\dot{y}=0 \tag{43}$$

$$y(0,t)=y''(0,t)=0 \text{ and } y(1,t)=y''(1,t)=0. \tag{44}$$

In this section, the dynamic response of elastic beam with one spring is analyzed for two spans. For M = 1, the resulting equation (43) becomes

$$y^{iv}+Py''+k\,\delta\left(x-\eta\right)y+\ddot{y}+\mu\dot{y}=0. \tag{45}$$

Thus, the operators given in the general model are as follows:

$$L_1[y]=y^{iv}+P_0y''+k\,\delta\left(x-\eta\right)y,\ L_2[y]=P_1y'',\ L_3[\dot{y}]=\mu\dot{y},\ a(x)=1,\ \Omega_1=\Omega \tag{46}$$

Substituting these terms to Eq. (10), one obtains

$$X_n^{iv}+P_0X_n''+k\,X_n\delta\left(x-\eta\right)-\omega_n^2X_n=0 \tag{47}$$

$$X_n(0)=X_n''(0)=0 \text{ and } X_n(1)=X_n''(1)=0 \tag{48}$$

Then, the coefficients (13) according to Eq. (47) are as follows:

$$b_{0,j}=1/\Delta x^4,\ b_{1,j}=\left(P_0\Delta x^2-4\right)/\Delta x^4,\ b_{2,j}=\left[6-2P_0\Delta x^2+\left(k\,\delta\left(x-\eta\right)-\omega_n^2\right)\Delta x^4\right]/\Delta x^4 \tag{49}$$

$$b_{3,j}=\left(P_0\Delta x^2-4\right)/\Delta x^4,\ b_{4,j}=1/\Delta x^4. \tag{50}$$

Here, all values except $b_{2,j}$ are constant for each the value of j. We substitute the coefficients (49,50) into the matrix (20). For nontrivial solution, the determinant of matrix of the coefficients should be equal to zero. Using this condition, the natural frequency of the system can be approximately found. The mode shape is also numerically obtained. Proceeding the perturbative solution, cases 1, 2, and 3 reveal this problem. Then, the coefficients α_{3n} and α_{2n} become

$$\alpha_{3n} = \frac{\mu}{2} \text{ and } \alpha_{2n} = -\frac{P_1}{4\omega_n}\int_0^1 X_n X_n'' \, dx \tag{51}$$

For calculating definite integral in the coefficient α_{2n}, Simpson's method is used. Then, the function f_j in the integral (51) is

$$f_j = \frac{X_{n,j}}{\Delta x^2}\left(X_{n,j+1} - 2X_{n,j} + X_{n,j-1}\right). \tag{52}$$

Comparing the obtained results, the same problem can be written as a set of equations of motions. Writing the equation in this form yields

$$y_1^{iv} + Py_1'' + \ddot{y}_1 + \mu\dot{y}_1 = 0 \tag{53}$$

$$y_2^{iv} + Py_2'' + \ddot{y}_2 + \mu\dot{y}_2 = 0 \tag{54}$$

with the boundary conditions $y_1(0,t) = y_1''(0,t) = 0$ and $y_2(1,t) = y_2''(1,t) = 0$ and the transient conditions $y_1 = y_2$, $y_1' = y_2'$, $y_1'' = y_2''$ and $y_1''' + Py_1' - ky_1 = y_2''' + Py_2'$.

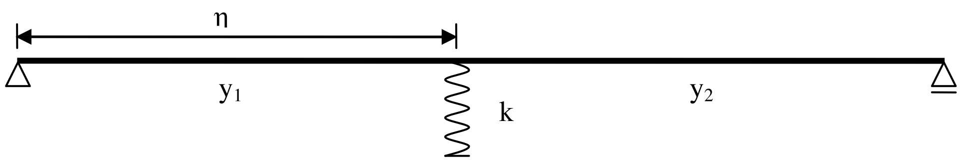

Figure 1. The elastic beam with one spring and two spans.

In Sınır & Sınır (2011), the coefficient α_{2n} is calculated as

$$\alpha_{2n} = -i\frac{P_1}{4\omega_n}\int_0^{\eta} X_{1n}X_{1n}'' \, dx + \int_{\eta}^{1} X_{2n}X_{2n}'' \, dx. \tag{55}$$

The comparison of the values of critical axially loading obtained for the coefficient of spring k and the location of spring η is given in the Table 1, where N denotes the number of grid points in the expansion of finite differences. Similarly, the values of the natural frequencies for the various locations of spring are given in Table 2.

Table 1. The comparison of axially critical loading obtained with Sınır & Sınır (2011) and the present method (bold) for $N = 200$.

k	η = 100		η = 1000	
0.1	11.6355	**11.6353**	18.6836	**18.6820**
0.3	20.4587	**20.4582**	30.7234	**30.7207**
0.5	29.2960	**29.2957**	39.4784	**39.4752**

Table 2. The comparison of the natural frequencies obtained with Sınır & Sınır (2011) and the present method (bold) for $P_0 = 10$ and $N = 200$.

k	$\eta = 100$		$\eta = 1000$	
0.1	4.0239	**4.0236**	9.7712	**9.7702**
0.3	10.4605	**10.4602**	19.5262	**19.5244**
0.5	13.8591	**13.8587**	34.1140	**34.1107**

Table 3. The comparison of the coefficients α_{2n} obtained with Sınır & Sınır (2011) and the present method (bold) for $P_0 = 10$, $P_1 = 1$ and $N = 200$.

k	$\eta = 100$		$\eta = 1000$	
0.1	-0.6150	**-0.6150**	-0.2778	**-0.2769**
0.3	-0.2466	**-0.2467**	-0.2212	**-0.2295**
0.5	-0.1792	**-0.1792**	-0.0984	**-0.0967**

Table 3 shows that the results obtained with classical method and finite differences are close to each other. It is seen that the numerical results and analytical solutions are almost the same.

THE BEAM-MASS SYSTEM

A beam-mass system consists of the Euler-Bernoulli beam and a concentrated mass on this beam. The equation of motion (Özkaya et al., 1997) according to proposed technique can be written as follows:

$$\left[\rho A + \bar{M}_j\,\delta\left(\hat{x} - x_j\right)\right]\ddot{\hat{w}} + EI\,\hat{w}^{iv} + \hat{\mu}\dot{\hat{w}} - \hat{g}\left(\hat{x}\right)\cos\hat{\Omega}\hat{t} = 0\,; \quad j = 1,2,\ldots,n \tag{56}$$

$$\hat{w}\left(0,\hat{t}\right) = \hat{w}''\left(0,\hat{t}\right) = 0 \text{ and } \hat{w}\left(1,\hat{t}\right) = \hat{w}''\left(1,\hat{t}\right) = 0. \tag{57}$$

where ρ is the density, and A is the cross-sectional area. $\bar{M}_j$ represents the concentrated mass. $\hat{\mu}$, $\hat{g}$ and $\hat{\Omega}$ denote viscous damping coefficient, the external excitation amplitude, and frequency. Introducing the dimensionless parameters

$$w = \hat{w}/L,\; x = \hat{x}/L,\; \eta_j = x_j/L,\; t = (\hat{t}/L^2)\sqrt{EI/\rho A} \tag{58}$$

$$\alpha_j = \bar{M}_j/\rho AL,\;\; \Omega = \hat{\Omega}L^2/\sqrt{EI/\rho A},\; \varepsilon g = \hat{g}L^3/EI,\; \varepsilon\mu = \hat{\mu}L^2/\sqrt{EI\rho A}, \tag{59}$$

then, the resulting equation is

$$\left[1 + \alpha_j\,\delta\left(x - \eta_j\right)\right]\ddot{w} + w^{iv} + \mu\dot{w} - g\left(x\right)\cos\Omega t = 0 \tag{60}$$

$$w\left(0,t\right) = w''\left(0,t\right) = 0 \text{ and } w\left(1,t\right) = w''\left(1,t\right) = 0. \tag{61}$$

where α is the ratio of the concentrated mass to the beam-mass. The operators corresponding to the general model are

$$L_1\left[y\right] = w^{iv},\; L_2\left[y\right] = 0,\; L_3\left[\dot{y}\right] = \mu\dot{w},\; a\left(x\right) = 1 + \alpha_j\delta\left(x - \eta_j\right),\; \Omega_2 = \Omega,\; F\left(x\right) = -g\left(x\right) \tag{62}$$

Then, Eq. (10) is obtained as

$$X_n^{iv} - \left(1 + \alpha_j \delta\left(x - \eta_j\right)\right)\omega_n^2 X_n = 0 \tag{63}$$

$$X_n(0) = X_n''(0) = 0 \text{ and } X_n(1) = X_n''(1) = 0. \tag{64}$$

The coefficients (13) from Eq. (63) are obtained as follows:

$$b_{0,j} = 1/\Delta x^4,\ b_{1,j} = -4/\Delta x^4,\ b_{2,j} = \left[6 - \omega_n^2\left(1 + \alpha_j \delta\left(x - \eta_j\right)\right)\Delta x^4\right]/\Delta x^4 \tag{65.a}$$

$$b_{3,j} = -4/\Delta x^4,\ b_{4,j} = 1/\Delta x^4 \tag{65.b}$$

Substituting the coefficients (63) to the matrix (20), the solution of the resulting matrix is smoothly obtained. Thus, the natural frequency of the system and the mode shape can be calculated. Cases 1 and 4 appear in this problem from the perturbative solution. Then, the coefficients α_{3n} and α_{1n} are as follows:

$$\alpha_{3n} = \frac{\mu}{2} \text{ and } \alpha_{1n} = -\frac{1}{2\omega_n}\int_0^1 g\left(x\right) X_n \, dx \tag{66}$$

The linear form of the mathematical models in (Özkaya et al., 1997 & Özkaya, 2001) is similar to the proposed equation in this study. In the result of performed calculations, the numerical comparisons are given in the following.

Table 4. The first three natural frequencies for different mass ratios and mass locations for one mass, where $N = 100$.

α	η	ω_1	**Özk.97**	ω_2	**Özk.97**	ω_3	**Özk.97**
1	0.0	9.8688	9.8695	39.4654	39.4784	88.7607	88.8264
	0.1	8.9954	8.9962	29.8755	29.8891	66.0088	66.0691
	0.2	7.4533	7.4541	26.9359	26.9462	73.4569	73.5140
	0.3	6.3941	6.3946	29.7397	29.7503	86.6638	86.7293
	0.4	5.8463	5.8468	35.2250	35.2374	79.9135	79.9788
	0.5	5.6791	5.6795	39.4654	39.4784	67.8305	67.8883
10	0.0	9.8688	9.8695	39.4654	39.4785	88.7607	88.8264
	0.1	5.3312	5.3322	19.8249	19.8959	59.0482	59.0995
	0.2	3.2594	3.2598	22.0495	22.0545	70.7174	70.7723
	0.3	2.5276	2.5279	26.7608	26.7706	86.0802	86.1462
	0.4	2.2250	2.2252	33.6682	33.6806	77.2029	77.2690
	0.5	2.1393	2.1395	39.4654	39.4785	62.3954	62.4517

Table 5. The natural frequencies corresponding to different mass ratios and mass locations for two masses, where $N = 100$.

α_1	α_2	η_1	η_2	ω_1	**Özk.01**	ω_2	**Özk.01**	ω_3	**Özk.01**	ω_4	**Özk.01**
1	1	0.1	0.3	6.118	6.118	27.536	26.506	55.338	55.412	98.966	99.097
			0.7	6.183	6.183	22.588	22.598	60.165	60.226	124.852	125.021
		0.5	0.3	4.730	4.785	25.116	19.802	60.830	45.252	141.073	95.238
			0.7	4.730	4.730	25.116	25.128	60.830	60.883	141.073	141.289
1	10	0.1	0.3	2.509	2.509	26.066	26.075	50.993	51.069	94.388	94.505
			0.7	2.516	2.516	20.052	20.060	58.763	58.824	124.117	124.285
		0.5	0.7	2.387	2.387	17.916	17.925	59.518	59.569	136.776	136.993
10	1	0.1	0.3	4.513	4.514	18.548	18.563	38.536	38.578	96.578	96.694
			0.7	4.671	4.671	12.423	12.429	50.941	50.992	121.270	121.432
		0.5	0.7	2.078	2.078	22.025	22.036	54.599	54.647	140.648	140.866
10	10	0.1	0.3	2.356	2.357	16.238	16.257	29.949	29.975	92.758	92.863
			0.7	2.412	2.413	8.845	8.850	48.883	48.883	120.858	121.018
		0.5	0.7	1.677	1.677	9.806	9.812	53.472	53.516	136.317	136.535

Table 6. The natural frequencies corresponding to different mass ratios and mass locations for three masses, where $N = 100$.

α_1	α_2	α_3	η_1	η_2	η_3	ω_1	**Özk.01**	ω_2	**Özk.01**	ω_3	**Özk.01**	ω_4	**Özk.01**
1	1	1	0.1	0.4	0.8	5.130	5.130	18.908	18.915	40.627	40.668	101.805	101.949
1	1	10	0.1	0.4	0.8	3.011	3.011	11.726	11.731	39.407	39.445	98.570	98.713
1	10	1	0.1	0.4	0.8	2.182	2.182	17.179	17.186	37.318	37.356	99.182	99.323
10	1	1	0.1	0.4	0.8	4.141	4.142	13.012	13.021	25.934	25.958	99.303	99.439
10	10	10	0.1	0.4	0.8	1.864	1.864	6.672	6.675	14.144	14.161	93.644	93.774
1	1	1	0.2	0.5	0.7	4.411	4.411	18.193	18.201	39.151	39.189	137.782	137.980
1	1	10	0.2	0.5	0.7	2.350	2.350	13.463	13.469	34.966	35.001	134.570	134.770
1	10	1	0.2	0.5	0.7	2.048	2.048	18.178	18.185	29.348	29.378	137.760	137.958
10	1	1	0.2	0.5	0.7	2.857	2.857	10.767	10.771	35.346	35.379	137.078	137.274
10	10	10	0.2	0.5	0.7	1.540	1.540	6.381	6.383	13.564	13.578	134.055	134.252

As shown in Table 4, the numerical results obtained for the natural frequencies are extremely close to analytical results. Similar findings are presented in Table 5 for two concentrated masses and in Table 6 for three masses.

CONCLUSIONS

In this study, a general model is considered to analyze the dynamic behavior of structural elements that may have variable material, cross-section, or some other discontinuities. Unlike the classical approach, instead of writing a separate equation for each span containing discontinuity, a single equation with singularity function is discussed. This provides great convenience to us in the solution, as all discontinuities occurring in any structural element are modeled with a single equation. To demonstrate the accuracy of the present technique, the general solution procedure has been applied to two different problems, such as the multilinear elastic spring beam and beam-mass system. As a result of the comparisons made, it has been observed that the results obtained as a result of applying the classical approach and the present method are extremely close to each other.

ACKNOWLEDGMENT

This work is supported by Manisa Celal Bayar University Scientific Research Project Coordination Unit (Project number: 2016-113).

REFERENCES

Beam deflections by discontinuity functions, IAST, Lect12.

Caddemi, S. & Morassi, A. 2013. Multi-cracked Euler-Bernoulli beams: mathematical modeling and exact solutions. International Journal of Solids and Structures, **50**: 944-956.

Caddemi, S., Caliò, I. & Cannizzaro, F. 2013a. Closed-form solutions for stepped Timoshenko beams with internal singularities and along-axis external supports. Arch. Appl. Mech. **83**: 559-577.

Caddemi, S., Caliò, I., Cannizzaro, F. & Rapicavoli, D. 2013b. A novel beam finite element with singularities for the dynamic analysis of discontinuous frames. Arch. Appl. Mech. **83**: 1451-1468.

Cheng, P., Davila, C. & Hou, G. 2014. Static, vibration analysis and sensitivity analysis of stepped beams using singularity functions. Journal of Structures, **2014**(15): 1-13.

Dinev, D. 2012. Analytical solution of beam on elastic foundation by singularity functions. Engineering Mechanics, **19**(5): 1-12.

Ghayesh, M.H., Kazemirad, S. & Reid, T. 2012. Nonlinear vibrations and stability of parametrically exited systems with cubic nonlinearities and internal boundary conditions: A general solution procedure. Applied Mathematical Modelling, **36**(7): 3299-3311.

Hosseini, S.A.A. & Zamanian, M. 2013. Analytical solution for general nonlinear continuous systems in a complex form. Applied Mathematical Modelling, **37**(3): 1163-1169.

Lacarbonara, W. 1999. Direct treatment and discretizations of non-linear spatially continuous systems. Journal of Sound and Vibration, **221**(5): 849-866.

Nayfeh, A.H. 1981. Introduction to Perturbation Techniques. Wiley, New York.

Sınır, B.G. & Sınır, S. 2011. Eksenel parametrik zorlamalı çok yaylı kirişlerin yeni bir yaklaşımla dinamik analizi, XVII. National Mechanical Congress, Fırat University, Elazığ, Turkey.

Sınır, B.G. 2015. Infinite mode analysis of a general model with external harmonic excitation. Applied Mathematical Modelling, **39**: 1823-1836.

Özkaya, E., Pakdemirli, M. & Öz, H.R. 1997. Non-Linear vibrations of a beam-mass system under different boundary conditions. Journal of Sound and Vibration, **199**(4): 679-696.

Özkaya, E. 2001. Linear transverse vibrations of a simply supported beam carrying concentrated mass. Mathematical and Computational Applications, **6**(3): 147-151.