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تأثير الداعم الدائري على الإهتزاز الحر للشرائح الإسطوانية ذات طبقات متدرجة الخواص مع طبقة وسطى من مواد متناظرة

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الخلاصة

في هذا التحليل للاهتزاز، قدمت دراسة عن تأثير الحلقات الداعمة للشرائح الأسطوانية ذات الطبقات متدرجة الخواص على هذة الشرائح التى تتكون من ثلاث طبقات ذات طبقة وسطى. طوقت الشرائح الأسطوانية بأستخدام الدواعم الحلقية. استُخدمت نظرية الباحث (لو ف) لإيجاد العلاقة بين الإستطالة والإنحناء. أستخدمت طريقة (رايل – رتز) لصياغة معادلة إهتزاز الشرائح الأسطوانية المكونة للقشرة. تم حساب الشكل المحوري للشرائح الأسطوانية باستخدام معدلات خصائص القضيب. نُفذت الدراسة بواسطة وضع الداعم الحلقي في مواقع مختلفة من الشرائح الأسطوانية المكونة للقشرة، وذلك لغرض معرفة الإهتزازات الطبيعية للشرائح الأسطوانية تحت تأثير حالات تثبيت مختلفة لعدد من العناصر الفيزيقية. تم التحقق من صحة النتائج بمقارنتها مع أعمال مشابهة منشورة سابقاً.

Influence of ring support on free vibration of sandwich functionally graded cylindrical shells with middle layer of isotropic material

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ABSTRACT

In this vibration analysis, influence of ring supports on functionally gradient three-layered sandwich cylindrical shells is presented with middle layer fabricated of isotropic material. The ring support is introduced beside radial direction of the shell. Love's first order thin shell theory is used for strain- and curvature-displacements relationship. Rayleigh–Ritz approach is employed to form the shell frequency equation. Axial modal dependence is approximated by characteristics beam functions. Study is carried out for placing ring support in different position of the shell, for different configuration of the functionally graded material's layers to investigate the natural frequencies of the cylindrical shells under different boundary conditions for a number of physical parameters. Results obtained are validated with the previous published works in the open literature.

Keywords: Three-layered sandwich cylindrical shell; functionally graded material; isotropic material; ring support; Love's thin shell theory; Rayleigh–Ritz approach.

INTRODUCTION

Amongst structural elements, shells have significant importance in engineering, particularly; cylindrical shells which are used massively in mechanical, aeronautical, marine and civil engineering. They have many applications in water tanks, large-span roofs, liquid containing bodies, nuclear power plants, piping system, pressure vessels technology, missiles, rockets, aircrafts, submarines and ships etc.

First attempt to solve the equations of cylindrical shells was made by Sophie Germane in 1821. Rayleigh (1882) suggested some simplifications in her work. The basic theoretical approach was given by Love (1888), who published the equations in their essential form at the end of the 19^{th} century. Love applied the Kirchhoff's hypothesis to design his shell theory, known as the first-order approximation theory. Most of the existing shell theories are based on it. Due to complex structure of shell, it has always been a challenge and the area of interest for researchers. Since then, a large number of developments in this field of vibrations of cylindrical shells were carried out.

Weingarten (1964) worked on vibration characteristics of finite shells. He obtained a frequency equation based on the known characteristic functions for beams with any combination of boundary conditions. Sewall and Naumann (1968) investigated experimentally natural frequencies of cylindrical shells with and without longitudinal stiffeners for different kinds of boundary conditions. The Rayleigh-Ritz method was employed to solve the shell problem. They used three types of shells in their investigation: an unstiffened shell, a shell with external longitudinal stiffeners and a third one with internal longitudinal stiffeners. Sharma and Johns (1971) gave a theoretical study of vibration characteristics of clamped-ring- stiffened and clamped-free shells. An approximate result was obtained by using Rayleigh-Ritz method. They used Flügge's shell theory and compared the results obtained from Timoshenko - Love Theory. Loy and Lam (1997) studied vibratory behaviour of shells implicating ring supports. They adopted Sander's shell theory to frame the shell problem. They examined the effects of boundary conditions on ring supported cylindrical shells. Wang *et al.* (1997) studied the vibration characteristics of varying ring–stiffener shells by using Ritz technique. They examined the influence of ring-stiffener eccentricity, location, number and their material distribution upon natural frequencies of the shells. Xiang *et al*. (2002) reported the exact vibration frequencies of shells for intermediate ring supports. They derived the system of homogeneous differential equations by making use of the state-space technique for a shell segment, while a domain decomposition technique was adopted for the sake of continuity among the segments of shell. Zhao *et al*. (2002) made an analytical study on vibrations of simply supported laminated cross-ply rotating cylindrical shells with ring stiffeners and stringer. The effects of the stiffeners were investigated by making use of the two methods. Firstly, a variational method where every stiffener has individual role and secondly, the averaging approach where characteristics of all stiffeners were averaged upon the surface of cylindrical shell. They concluded that averaging method was sensitive and produced inaccurate results whereas the variational formulation was fast and better. Xiang *et al.* (2005) presented the elastic buckling of axially dense ring supported shells placed at intermediate position. They used the Timoshenko thin shell theory and Flügge shell theory to analyze the buckling solutions of shell.

Shells are constructed with isotropic as well as composite materials. Among composite materials, functionally graded material (FGM) is one whose compositions and functionality vary constantly smoothly through the thickness. Loy *et al*. (1999) first analyzed free vibrations of functionally graded cylindrical shells fabricated from nickel and stainless steel. They studied the influence of the configurations of constituent materials upon the shell's natural frequencies by using Love's shell theory and Rayleigh-Ritz numerical approach. Pradhan *et al.* (2000) extend their work for different kinds of end conditions. Najafizadeh and Isvandzibaei (2007) analyzed vibration of FGM circular cylindrical shells. They employed third order shear deformation plate theory to study effects of ring support upon the natural frequencies of cylindrical shells. Isvandzibaei and Awasare (2010) also investigated the vibration characteristics of functionally graded shell with ring supports. They used Hamilton's principle and third order deformation shear theory for free-free end. Arshad *et al*. (2010) explained frequency behaviour of bi-layered cylindrical shells. One of the layers was fabricated from FGM while the other layer was composed of isotropic material. They studied the effect of various shell configurations upon the shell's natural frequencies. They later extended their work for the vibration characteristics of cylindrical shells fabricated with two independent functionally graded material layers (Arshad *et al*.2011). Li *et al*. (2010) analyzed vibration characteristics for three layered simply supported cylindrical shells whose inner and outside layers were fabricated with the same homogeneous isotropic material whereas the middle layer consisted of an FGM. Flügge's shell theory was used to derive governing shell equations. They found that the natural frequency decreased by increasing thickness to radius ratio, whereas it increased by increasing Young's modulus ratio at mid shell surface to that of the outer or the internal shell layer.

In the present study, three-layered cylindrical shell is considered for investigation, in which the inner and outer layers are composed of FGMs, whereas the middle layer is assumed to be made of isotropic material. With the induction of ring support in radial direction, the ring support influence is introduced. Effects of thickness configurations of the layers and by placement of ring support in different positions, are also examined. An eigenvalue problem is framed by employing Rayleigh-Ritz formulation. Axial modal fields are measured by characteristics beam functions derived from the beam equation for various boundary conditions.

FORMATION OF A SHELL PROBLEM

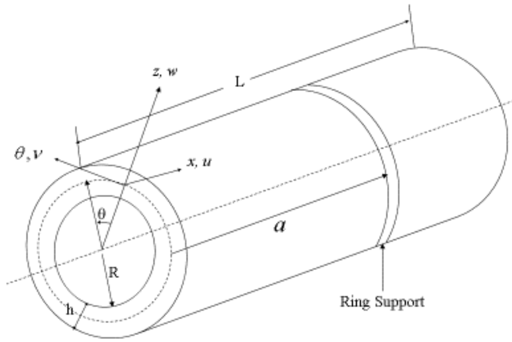

Fig. 1. Geometry of Cylindrical shell with ring support

Consider a thin circular cylindrical shell enclosed by a ring support as shown in Figure 1, R and L represent radius and length of the cylindrical shell respectively. h stands for shell thickness and a_i denotes position of i^{th} ring support in longitudinal direction of circular shell. An orthogonal coordinate system (x, θ, z) is established at the shell mid surface, while u, v, w are mid surface deformation displacements in axial, circumferential and radial directions x, θ, z, respectively. Applying condition of plane stress, constitutive relations of a circular shell according to Hooke's law are written as:

$$\{\sigma\} = [Q]\{e\} \tag{1}$$

where $\{\sigma\}$ and $\{e\}$ are the stress and strain vectors. $[Q]$ stands for reduced stiffness matrix. The stress and strain vectors are defined as:

$$(\sigma)^T = (\sigma_x, \sigma_\theta, \sigma_{x\theta}) \tag{2}$$

$$(e)^T = (e_x, e_\theta, e_{x\theta}) \tag{3}$$

where σ_x and σ_θ represent normal stresses in the x - and θ -directions, respectively and $\sigma_{x\theta}$ denotes shear stress in $x\theta$ -plane. Similarly e_x and e_θ are the normal strains in the x -and θ -directions and $e_{x\theta}$ is the shear strain along the $x\theta$ -plane. The expression for reduced stiffness matrix $[Q]$ is defined as:

$$Q = \begin{pmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{pmatrix} \tag{4}$$

So, the equation (1) can be re-written as:

$$\begin{pmatrix} \sigma_x \\ \sigma_\theta \\ \sigma_{x\theta} \end{pmatrix} = \begin{pmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{pmatrix} \begin{pmatrix} e_x \\ e_\theta \\ e_{x\theta} \end{pmatrix} \tag{5}$$

For isotropic materials, the reduced stiffness Q_{ij} (i, j= 1, 2 and 6) are expressed as:

$$\left(Q_{11},\ Q_{12},\ Q_{66}\right) = \left(\frac{E}{1-\upsilon^2},\ \frac{\upsilon E}{1-\upsilon^2},\ \frac{E}{2\left(1+\upsilon\right)}\right),\ Q_{22} = Q_{11} \tag{6}$$

where, E represents Young's modulus and υ stands for Poisson's ratio. The strain components of strain vector $\{e\}$ are written in terms of the thickness variable z with linear expressions:

$$\left(e_x, e_\theta, e_{x\theta}\right) = \left(e_1 + z\kappa_1, e_2 + z\kappa_2, \gamma + 2z\tau\right) \tag{7}$$

where e_1, e_2 and γ are reference surface strains and κ_1, κ_2 and τ are surface curvatures. There are a number of shell theories found in open literature on shell problem. For the present case, the formulae for reference surface strains and curvatures are adopted from Love's first order shell theory for simplicity. These surface strains and curvatures are given as:

$$\{e_1, e_2, \gamma\} = \left\{\frac{\partial u}{\partial x}, \frac{1}{R}\left(\frac{\partial v}{\partial \theta} + w\right), \frac{\partial v}{\partial x} + \frac{1}{R}\frac{\partial u}{\partial \theta}\right\} \tag{8}$$

$$\{\kappa_1, \kappa_2, \tau\} = \left\{-\frac{\partial^2 w}{\partial x^2}, -\frac{1}{R^2}\left(\frac{\partial^2 w}{\partial \theta^2} - \frac{\partial v}{\partial \theta}\right), -\frac{1}{R}\left(\frac{\partial^2 w}{\partial x \partial \theta} - \frac{\partial v}{\partial x}\right)\right\} \tag{9}$$

The force and moment resultants for a cylindrical shell are defined as:

$$(N_x, N_\theta, N_{x\theta}) = \int_{-h/2}^{h/2} (\sigma_x, \sigma_\theta, \sigma_{x\theta}) dz \tag{10a}$$

$$(M_x, M_\theta, M_{x\theta}) = \int_{-h/2}^{h/2} (\sigma_x, \sigma_\theta, \sigma_{x\theta}) z dz \tag{10b}$$

where N_x, N_θ and $N_{x\theta}$ are the force resultants in the axial, circumferential and shear directions, respectively. Similarly M_x, M_θ and $M_{x\theta}$ denote moment resultants in the same directions, respectively. From the Equations (5), (7) and (10), we have:

$$[N]=[S]\ [\varepsilon] \tag{11}$$

where $[N]$ and $[\varepsilon]$ stand for,

$$[N]^T=[N_x, N_\theta, N_{x\theta}, M_x, M_\theta, M_{x\theta}] \tag{12}$$

$$[\varepsilon]^T=[e_1, e_2, \gamma, \kappa_1, \kappa_2, 2\tau] \tag{13}$$

respectively, and $[S]$ is defined as:

$$[S]=\begin{bmatrix} A_{11} & A_{12} & 0 & B_{11} & B_{12} & 0 \\ A_{12} & A_{22} & 0 & B_{12} & B_{22} & 0 \\ 0 & 0 & A_{66} & 0 & 0 & B_{66} \\ B_{11} & B_{12} & 0 & D_{11} & D_{12} & 0 \\ B_{12} & B_{22} & 0 & D_{12} & D_{22} & 0 \\ 0 & 0 & B_{66} & 0 & 0 & D_{66} \end{bmatrix} \tag{14}$$

Here $[A]$, $[B]$ and $[D]$ represent the sub-matrices of extensional, coupling and bending stiffnesses, respectively and are defined as:

$$[A_{ij}, B_{ij}, D_{ij}]=\int_{-h/2}^{h/2} Q_{ij}[1, z, z^2]\ dz \tag{15}$$

The coupling stiffness B_{ij} reduces to zero for isotropic cylindrical shell whereas for a non-homogeneous functionally gradient, cylindrical shell B_{ij} is involved and is non-zero. The reason for this is that the material properties are asymmetric about the mid plane, when Q_{ij} depend on the thickness variable z. Strain energy, denoted by U for a free vibrating cylindrical shell is expressed as:

$$U=1/2\int_0^L\int_0^{2\pi}[\varepsilon]^T[S]\ [\varepsilon]\ Rd\theta\ dx \tag{16}$$

Making substitutions for $[\varepsilon]^T$, $[S]$ and $[\varepsilon]$ from Equations. (13) and (14) into Equation (16), new form of the shell strain energy U, is given by:

$$U=\frac{R}{2}\int_0^L\int_0^{2\pi}[A_{11}e_1^2+A_{22}e_2^2+2A_{12}e_1e_2+A_{66}\gamma^2+2B_{11}e_1\kappa_1+2B_{12}e_1\kappa_2+2B_{12}e_2\kappa_1$$

$$+2B_{22}e_2\kappa_2+4B_{66}\gamma\tau+D_{11}\kappa_1^2+D_{22}\kappa_2^2+2D_{12}\kappa_1\kappa_2+4D_{66}\tau^2]d\theta\ dx \tag{17}$$

Substituting relations of strains and curvatures from Equations (8) and (9) into Equation (17), we get:

$$U = \frac{R}{2}\int_0^L\int_0^{2\pi}\Bigg[A_{11}\left(\frac{\partial u}{\partial x}\right)^2 + \frac{A_{22}}{R^2}\left(\frac{\partial v}{\partial \theta} + w\right)^2 + \frac{2A_{12}}{R}\left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial v}{\partial \theta} + w\right) + A_{66}\left(\frac{\partial v}{\partial x} + \frac{1}{R}\frac{\partial u}{\partial \theta}\right)^2$$
$$-2B_{11}\left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial^2 w}{\partial x^2}\right) - \frac{2B_{12}}{R^2}\left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial^2 w}{\partial \theta^2} - \frac{\partial v}{\partial \theta}\right) - \frac{2B_{12}}{R}\left(\frac{\partial v}{\partial \theta} + w\right)\left(\frac{\partial^2 w}{\partial x^2}\right)$$
$$-\frac{2B_{22}}{R^3}\left(\frac{\partial v}{\partial \theta} + w\right)\left(\frac{\partial^2 w}{\partial \theta^2} - \frac{\partial v}{\partial \theta}\right) - \frac{4B_{66}}{R}\left(\frac{\partial v}{\partial x} + \frac{1}{R}\frac{\partial u}{\partial \theta}\right) - \frac{4B_{66}}{R}\left(\frac{\partial v}{\partial x} + \frac{1}{R}\frac{\partial u}{\partial \theta}\right)$$
$$\left(\frac{\partial^2 w}{\partial x \partial \theta} - \frac{\partial v}{\partial x}\right) + D_{11}\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + \frac{D_{22}}{R^4}\left(\frac{\partial^2 w}{\partial \theta^2} - \frac{\partial v}{\partial \theta}\right)^2 + \frac{2D_{12}}{R^2}\left(\frac{\partial^2 w}{\partial x^2}\right)$$
$$\left(\frac{\partial^2 w}{\partial \theta^2} - \frac{\partial v}{\partial \theta}\right) + \frac{4D_{66}}{R^2}\left(\frac{\partial^2 w}{\partial x \partial \theta} - \frac{\partial v}{\partial x}\right)^2\Bigg] d\theta\, dx \tag{18}$$

Energy of a system consists of two basic forms by virtue of its position and motion, i.e., strain and kinetic energy respectively. Strain energy of a cylindrical shell has been described in the Equation (18) above, whereas its counterpart, i.e., kinetic energy, symbolized by *T*, comprising three physical basic quantities, mass, velocity and time, is given as:

$$T = \frac{R}{2}\int_0^L\int_0^{2\pi} \rho_T \left[\left(\frac{\partial u}{\partial t}\right)^2 + \left(\frac{\partial v}{\partial t}\right)^2 + \left(\frac{\partial w}{\partial t}\right)^2\right] d\theta\, dx \tag{19}$$

where ρ_T is mass density found in unit length and is defined as:

$$\rho_T = \int_{-h/2}^{h/2} \rho dz$$

where ρ is mass density. The Lagrangian energy functional Π of a circular cylindrical shell is expressed as:

$$\Pi = T - U \tag{20}$$

MODAL DISPLACEMENT DEFORMATIONS

For linear systems of partial differential equations, the method of separation of variables is generally employed. For the present situation, the partial differential equation system involves three unknown functions *u*, *v*, *w*, which denote the shell deformation in axial, circumferential and radial directions, respectively. Their modal displacement shapes are taken to be as follows:

$$u(x,\theta,t) = A_m \frac{d\phi}{dx}\cos n\theta \sin \omega t \tag{21a}$$

$$v(x,\theta,t)=B_m\phi(x)\sin n\theta\sin\omega t \tag{21b}$$

$$w(x,\theta,t)=C_m\varphi(x)\prod_{i=1}^{N}(x-a_i)^{\varepsilon_i}\cos n\theta\sin\omega t \tag{21c}$$

where A_m, B_m and C_m are constants which denote the vibration amplitudes in the x, θ and z directions, respectively. The axial function $\phi(x)$ represents axial modal displacement shapes and satisfies the geometric boundary conditions; n denotes the circumferential wave number and ω is circular vibration frequency. The i^{th} ring support is at $x = a_i$ along the longitudinal direction. N is the number of ring supports. The parameter ε_i shows the existence of ring supports. When $\varepsilon_i = 1$, the ring supports are present, when ε_i is zero, no ring support is applied on the shell.

FORMATION OF FREQUENCY EQUATION

Substituting Equation (21) for modal deformation displacements into the strain and kinetic energy expressions (19) and (20) for the frequency equations of the cylindrical shell. These expressions can be used for any number of ring support but for simplicity, only one ring support effect is investigated. Now, applying the principle of minimization of the energy, the expressions for maximum strain energy $U_{\max}$ and maximum kinetic energy $T_{\max}$ are obtained, i.e.,

$$U_{\max}=\frac{\pi R}{2}\int_0^L\Bigg[A_{11}A_m^2\left(\frac{d^2\phi}{dx^2}\right)^2+\frac{n^2A_{22}}{R^2}B_m^2\phi^2+\frac{A_{22}}{R^2}C_m^2(x-a)^2\phi^2+\frac{2nA_{22}}{R^2}B_mC_m(x-a)\ \phi^2$$

$$+\frac{2nA_{12}}{R}A_mB_m\phi\frac{d^2\phi}{dx^2}+\frac{2A_{12}}{R}A_mC_m(x-a)\phi\frac{d^2\phi}{dx^2}+A_{66}\left(B_m-\frac{n}{R}A_m\right)^2\left(\frac{d\phi}{dx}\right)^2$$

$$-2B_{11}A_mC_m(x-a)\left(\frac{d^2\phi}{dx^2}\right)^2-4B_{11}A_mC_m\frac{d\phi}{dx}\frac{d^2\phi}{dx^2}+\frac{2n^2B_{12}}{R^2}A_mC_m(x-a)$$

$$\phi\frac{d^2\phi}{dx^2}+\frac{2nB_{12}}{R^2}A_mB_m\phi\frac{d^2\phi}{dx^2}-\frac{2nB_{12}}{R}B_mC_m(x-a)\phi\frac{d^2\phi}{dx^2}-\frac{4nB_{12}}{R}B_mC_m\phi\frac{d\phi}{dx}$$

$$-\frac{2B_{12}}{R}C_m^2(x-a)^2\phi\frac{d^2\phi}{dx^2}-\frac{4B_{12}}{R}C_m^2(x-a)\phi\frac{d\phi}{dx}+\frac{2B_{22}}{R^3}(n^3B_mC_m(x-a)\phi^2$$

$$+n^2B_m\phi^2+n^2C_m^2(x-a)^2\phi^2+nB_mC_m(x-a)^2\phi)+\frac{4B_{66}}{R}\Bigg[nB_mC_m\left(\frac{d\phi}{dx}\right)^2$$

$$(x-a)+nB_mC_m\phi\frac{d\phi}{dx}+B_m^2\left(\frac{d\phi}{dx}\right)^2-\frac{n^2}{R}A_mC_m\left(\frac{d\phi}{dx}\right)^2(x-a)-\frac{n^2}{R}A_mC_m\phi\frac{d\phi}{dx}$$

$$-\frac{n}{R}A_mB_m\left(\frac{d\phi}{dx}\right)^2\Bigg]+D_{11}C_m^2\left[\left(\frac{d^2\phi}{dx^2}\right)^2(x-a)^2+4\left(\frac{d\phi}{dx}\right)^2+4(x-a)\frac{d\phi}{dx}\frac{d^2\phi}{dx^2}\right]+\frac{D_{22}}{R^4}$$

$$\left[n^4C_m^2(x-a)^2\phi^2+n^2\phi^2B_m^2+2n^3B_mC_m(x-a)\phi^2\right]-\frac{2D_{12}}{R^2}\Bigg[n^2C_m^2\frac{d^2\phi}{dx^2}$$

$$(x-a)^2\phi+nB_mC_m(x-a)\phi\frac{d^2\phi}{dx^2}+2n^2C_m^2(x-a)\phi\frac{d\phi}{dx}+2nB_mC_m\phi\frac{d\phi}{dx}\Bigg]$$

$$+\frac{4D_{66}}{R^2}\Bigg[n^2C_m^2(x-a)^2\left(\frac{d\phi}{dx}\right)^2+n^2C_m^2\phi^2+2n^2C_m^2\phi\frac{d\phi}{dx}+B_m^2\left(\frac{d\phi}{dx}\right)^2+2nB_mC_m$$

$$(x-a)\left(\frac{d\phi}{dx}\right)^2+2nB_mC_m\phi\frac{d\phi}{dx}\Bigg]\Bigg]\quad dx \tag{22}$$

and

$$T_{\max}=\frac{\pi R\rho_t\omega^2}{2}\int_0^L\left[A_m^2\left(\frac{d\phi}{dx}\right)^2+B_m^2\phi^2+C_m^2\phi^2(x-a)^2.\right]dx \tag{23}$$

Energy functional Π defined by the Lagrangian functional takes the following form:

$$\Pi=T_{\max}-U_{\max} \tag{24}$$

Amongst the different numerical methods the Rayleigh-Ritz method is considered to be the best to solve the shell problem as a variational approach. This technique has been used by several researchers to investigate the vibration characteristics of cylindrical shells by making use of the energy functional by extremizing it. The functional is minimized with regard to the vibration amplitudes A_m, B_m and C_m as:

$$\frac{\partial\Pi}{\partial A_m}=\frac{\partial\Pi}{\partial B_m}=\frac{\partial\Pi}{\partial C_m}=0 \tag{25}$$

This generates three homogeneous algebraic equations which leads to the eigenvalue problem and can be written in matrix notation as follows:

$$\begin{bmatrix}a_{11}&a_{12}&a_{13}\\a_{12}&a_{22}&a_{23}\\a_{13}&a_{23}&a_{33}\end{bmatrix}\begin{bmatrix}A_m\\B_m\\C_m\end{bmatrix}=\rho_t\omega^2\begin{bmatrix}I_2&0&0\\0&I_9&0\\0&0&I_{11}\end{bmatrix} \tag{26}$$

The expressions for the terms a_{ij}'s, I_2, I_9 and I_{11} are given in appendix-I. MATLAB package is used to solve the eigenvalue Equation (26) for investigating shell vibrations.

It provides an interactive working environment to carry out the quite complex computational tasks with a few commands. With the help of a simple command, one gets both the eigenvalues representing shell natural frequencies and the eigenvectors that correspond to the mode shapes.

BOUNDARY CONDITIONS

Boundary conditions are those constraints prescribed to study the behavior of the solution of differential equations at boundary of solution domain. They have significant importance for determining the analytical solutions of many physical problems. The geometric boundary conditions are classified as: clamped, free and simply supported and written mathematically in terms of $\phi(x)$ as:

i) Clamped boundary condition:

$$\phi(x) = \phi'(x) = 0 \tag{27a}$$

ii) Free boundary condition:

$$\phi''(x) = \phi'''(x) = 0 \tag{27b}$$

iii) Simply supported boundary condition:

$$\phi(x) = \phi''(x) = 0 \tag{27c}$$

AXIAL MODAL FUNCTIONS

There are different types of mathematical functions which are chosen to measure the axial modal dependence. In usual practice, characteristic beam functions, Ritz functions and exponential functions are utilized to achieve this end. The axial function $\phi(x)$ being beam function is:

$$\phi(x) = \beta_1 \cosh\left(\frac{\mu_m x}{L}\right) + \beta_2 \cos\left(\frac{\mu_m x}{L}\right) - \xi_m \left(\beta_3 \sinh\left(\frac{\mu_m x}{L}\right) + \beta_4 \sin\left(\frac{\mu_m x}{L}\right)\right) \tag{28}$$

where $\beta_i\,(i = 1,4)$ are constants having value zero or one depending on a boundary condition. μ_m denote the roots of some transcendental equations and ξ_m are parameters which are functions of μ_m.

NUMERICAL RESULTS AND DISCUSSION

A few comparisons of numerical results are done to check the validity, accuracy and robustness of numerical procedure for simply supported - simply supported (SS-SS), clamped - clamped (C-C) and clamped-free (C–F) end conditions. For this purpose, shell vibration frequencies for isotropic and functionally graded cylindrical shells are compared with those results found in the open literature.

In Table 1, non-dimensionalized frequency parameters $\Omega = \omega R\sqrt{(1-\upsilon^2)\rho / E}$ of an isotropic cylindrical shell are compared with those determined by Xiang *et al.* (2002) for simply supported boundary conditions applied on both shell ends. The shell parameters are given in the table. The axial wave mode is assumed to be unity and circumferential wave numbers (n) are selected from 1 to 4. It is observed that the two sets of results agreed very well with each other as the percentage difference is negligible.

In Table 2, the variation of frequency parameters $\Omega = \omega R\sqrt{(1-\upsilon^2)\rho / E}$ of isotropic cylindrical shell with circumferential wave number *n* (=1, 2, 3, 4) for clamped ends are compared with those calculated by Zhang *et al.* (2001). Axial half wave mode, *m*=1. They presented vibration characteristics of circular shells by applying wave propagation method. Finite element method was utilized for comparisons of results. They concluded that the wave propagation method was convenient, effective and accurate. The comparison shows an excellent agreement between two sets of frequency parameters.

Table 1. Comparison of frequency parameter $\Omega = \omega R\sqrt{(1-\upsilon^2)\rho / E}$ of a simply supported isotropic cylindrical shell ($m=1, L/R = 20, h/R = 0.05, E = 30\times10^6$ Ibf in^{-2}, $\upsilon = 0.3, \rho = 7.35\times10^{-4}$ Ibf s^2 in^{-4}).

n	Xiang *et al.* [8]	Present	%Difference
1	0.0161029	0.016103	-0.16
2	0.0392710	0.039271	0.00
3	0.1098116	0.10981	-0.001
4	0.2102773	0.21028	0.001

Table 2. Comparison of frequency parameter $\Omega = \omega R\sqrt{(1-\upsilon^2)\rho / E}$ of an isotropic cylindrical shell with C-C end conditions ($m=1, L/R = 20, h/R = 0.01, \upsilon = 0.3$).

n	Zhang *et al.* [18]	Present	%Difference
1	0.03487	0.034395	-1.36
2	0.014052	0.014256	1.45
3	0.022725	0.022713	-0.05
4	0.042271	0.042216	-0.13
5	0.068116	0.068051	-0.09
6	0.099823	0.099755	-0.06
7	0.137328	0.13726	-0.04
8	0.180617	0.18055	-0.03
9	0.226984	0.22961	1.15
10	0.284526	0.28445	-0.02

Table 3. Comparison of frequency parameter $\Omega = \omega R\sqrt{(1-\upsilon^2)\rho / E}$ of an isotropic cylindrical shell with clamped - free condition ($m=1$, $L = 254$ mm, $h = 1.5$ mm, $R = 32.25$ mm, $E = 207\times10^9$ N/m^2, $\upsilon = 0.28$, $\rho = 7.86\times10^3$ kg/m^3).

n	Naeem *et al.* [19]	Present	%Difference
1	0.0348	0.0372	6.89
2	0.0381	0.0385	1.04
3	0.1022	0.1026	0.39
4	0.1954	0.1959	0.25

In Table 3, a comparison of frequency parameter $\Omega = \omega R\sqrt{(1-\upsilon^2)\rho / E}$ for a clamped-free cylindrical shell is performed with those determined by Naeem *et al.* (2009). They presented a study on frequency characteristics of functionally graded material cylindrical shells by applying the Ritz formulation. To obtain a rapid convergence of the method, Ritz polynomials were used to approximate axial modal dependence. The half axial wave mode is taken to be $m=1$, whereas circumferential wave numbers *n* were taken from 1 to 4. From comparison, it is perceived that a close agreement is found between the two sets of frequencies. From above comparisons of different sets of results obtained by utilizing present technique, for three end conditions, it is obvious that present results show the validity, accuracy and robustness of the Rayleigh-Ritz method.

Frequency parameter variation versus circumferential wave numbers (*n*) with a ring support

Vibrations of cylindrical shells supported with a ring support at arbitrary position along the shell circumferential direction is investigated and this is done by setting $\xi_i = 1$ in Equation (21c) for three boundary conditions viz., simply supported - simply supported (SS-SS), clamped-free (C-F) and clamped-clamped (C-C). Figure 2 depicts the variations of frequency parameter $\Omega = \omega R\sqrt{(1-\upsilon^2)\rho / E}$ for isotropic cylindrical shells with a ring support attached at $a = 0.3L$ with circumferential wave number *n*, whereas the axial wave mode m is taken to be 1, 2 and 3 for three end conditions: SS-SS, C-C and C–F respectively. Frequency parameters increased by increasing the circumferential wave number *n* as shown in Figures 2(a)-(c).

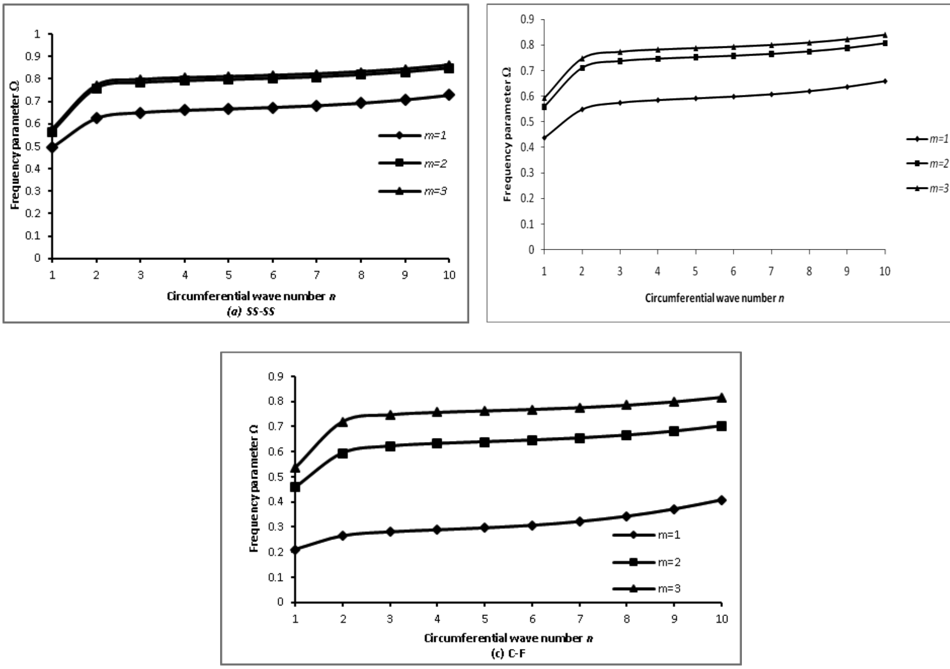

Fig. 2. Frequency parameter $\Omega = \omega R\sqrt{(1-\upsilon^2)\rho / E}$ variation versus circumferential wave numbers (n), of a (a) SS – SS, (b) C-C, (c) C-F cylindrical shells with a ring support ($L/R = 20, h/R = 0.01, a/L = 0.3$)

As n goes from 1 to 2 for three boundary conditions, the increase in frequency parameter is the most noticeable. It can be seen that vibration frequencies gradually increase for n greater than 2 with circumferential wave number. Frequency pattern shows that fundamental frequency of a ring supported cylindrical shell corresponds to $n=1$. Shell frequencies increase with increasing values of m but minimum frequency is associated with $n=1$ for three values of m. It is evident from the figures that frequencies of the shells for axial wave numbers $m=2, 3$ are very close to each other for simply supported boundary condition as compared to other two boundary conditions. Similarly, for C-F boundary condition, natural frequencies of the shells are far apart from each other for the axial wave numbers $m=1, 2, 3$ than the other two boundary conditions.

THREE LAYERED CYLINDRICAL SHELLS

Numerical results for three-layered configuration for cylindrical shells are investigated for various physical parameters. A cylindrical shell configuration is considered to be composed of three layers along radial thickness direction. Out of these three layers, material of the central layer is of isotropic nature whereas, the inside and outside layers are composed of FGM. The cross-sectional vision of three layered cylindrical shell is shown in the following Fig. 3.

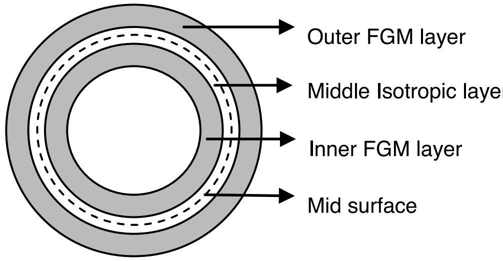

Fig. 3. Cross-sectional vision of three layered cylindrical shell

In this work, zirconia and stainless steel are used for fabrication FGM. Nickel being isotropic is used for the middle layer. Material parameters of shell are effective Young's modulus (E), Poisson's ratio (υ) and mass density (ρ). Young's modulus has the most significant influence on the shell vibrating characteristics. Effect of variable thickness of middle isotropic layer is examined. Here, the stiffness modulii are modified by the variation of material thickness distribution such as:

$$\left.\begin{aligned} A_{ij} &= A_{ij}^{in(FGM)} + A_{ij}^{mid(iso)} + A_{ij}^{out(FGM)} \\ B_{ij} &= B_{ij}^{in(FGM)} + B_{ij}^{mid(iso)} + B_{ij}^{out(FGM)} \\ D_{ij} &= D_{ij}^{in(FGM)} + D_{ij}^{mid(iso)} + D_{ij}^{out(FGM)} \end{aligned}\right] \tag{29}$$

where $i, j = 1, 2, 6$ and superscript *in(FGM), out(FGM)* represent the inner and outer functionally graded layers respectively and *mid(iso)* is associated with middle isotropic layer of a cylindrical shell. If we take M_1, M_2 constituent materials at the inner FGM layer and M_3, M_4 at the outer FGM layer, the resultant material properties Young's modulii, Poisson ratios and mass density of inner and outer FGM layers are given as:

$$\left.\begin{aligned} E^{in(FGM)} &= \left(E_2 - E_1\right)\left(3\frac{z}{h} + \frac{3}{2}\right)^p + E_1 \\ \nu^{in(FGM)} &= \left(\nu_2 - \nu_1\right)\left(3\frac{z}{h} + \frac{3}{2}\right)^p + \nu_1 \\ \rho^{in(FGM)} &= \left(\rho_2 - \rho_1\right)\left(3\frac{z}{h} + \frac{3}{2}\right)^p + \rho_1 \end{aligned}\right] \tag{30}$$

$$\left.\begin{aligned} E^{out(FGM)} &= \left(E_4 - E_3\right)\left(3\frac{z}{h} - \frac{1}{2}\right)^p + E_3 \\ \nu^{out(FGM)} &= \left(\nu_4 - \nu_3\right)\left(3\frac{z}{h} - \frac{1}{2}\right)^p + \nu_3 \\ \rho^{out(FGM)} &= \left(\rho_4 - \rho_3\right)\left(3\frac{z}{h} - \frac{1}{2}\right)^p + \rho_3 \end{aligned}\right] \tag{31}$$

Thickness of the layers is assumed to be equally distributed then it is evident from Equation (30) that at *z* = *-h/2,* the resultant material properties attain the configurations of constituent material M_1 and for *z* = *-h/6*, the resultant material properties reach to the configuration of constituent material M_2. Similarly, in Equation (31), at the outer FGM layer for *z* = *+h/6*, the resulting material properties are of constituent material M_3 whereas for *z* = *+h/2,* at the outer surface of the outer FGM layer, material parameters tend to that of constituent material M_4. For simplicity, take the same FGM constituents at the inner and outer functionally graded layers and by changing the order of functionally graded constituent materials and by keeping the same material for isotropic middle layer, the four types of cylindrical shells can be formed, which are mentioned in Table 4.

Table 4. Types of Shell w.r.t. configurations

Types	Inner FGM layer		Middle Isotropic layer	Outer FGM layer	
	Inner surface	**Outer surface**		**Inner surface**	**Outer surface**
Shell I	Stainless Steel	Zirconia	Nickel	Stainless Steel	Zirconia
Shell II	Stainless Steel	Zirconia	Nickel	Zirconia	Stainless Steel
Shell III	Zirconia	Stainless Steel	Nickel	Stainless Steel	Zirconia
Shell IV	Zirconia	Stainless Steel	Nickel	Zirconia	Stainless Steel

The material parameters for stainless steel, zirconia and nickel evaluated at the temperature T=300K, are described by Loy *et al*. (1999) and Pradhan *et al.* (2000). Different pattern of thickness of the layers of cylindrical shell is presented in Table 5.

Table 5. Variation of thickness of shell layers

Thickness Configurations	Inner FGM Layer	Middle Isotropic Layer	Outer FGM Layer
Case 1	$5h/12$	$h/6$	$5h/12$
Case 2	$h/4$	$h/4$	$h/2$
Case 3	$h/4$	$h/2$	$h/4$
Case 4	$h/6$	$2h/3$	$h/6$

Effect of shell configurations on vibration frequencies (Hz) versus circumferential wave numbers (*n*) without ring supports

In Tables 6-8, the natural frequencies for the shells type I, type II, type III and type IV are presented against the circumferential wave numbers (*n*) versus power law exponents *p* =3, 5 and 7 for three boundary conditions namely: simply supported - simply supported (SS–SS), clamped–free (C–F) and clamped–clamped (C–C). Shell geometrical parameters are assumed to be: $L/R = 2, h/R = 0.05$ and the axial half wave mode is *m* =1. The thickness of each layer is supposed to be $h/3$. For all four types of cylindrical shells, it is seen from these tables that the frequencies first decrease with the increasing values of circumferential wave number n=3 and then start to increase with the ascending value of n. It is also noted that natural frequencies increase by increasing power law exponent (p). It is also observed, that the frequencies for cylindrical shells Type II and Type III fall between the frequencies of shell type I and type IV.

It might be due to the same configuration of FG constituents at the inner and outer FGM layers in the shells type I & type IV while the configuration of functionally graded constituents in shell type II is the inverse of the shell type III at the inner and outer FGM layers, respectively. It is also observed that the configuration of the FGM constituents are symmetric in shells type II and type III.

Table 6. Natural frequencies (Hz) variation versus circumferential wave numbers (n), of SS-SS cylindrical shells ($m=1, L/R=2, h/R=0.05$).

p	***n***	**Shell I**	**Shell II**	**Shell III**	**Shell IV**
3	1	596.97	583.55	583.82	572.08
	2	347.74	339.80	340.08	333.11
	3	246.77	240.60	242.46	237.03
	4	276.45	268.63	273.64	266.74
5	1	617.99	597.57	597.67	580.71
	2	359.95	347.98	348.12	338.17
	3	255.22	246.54	247.98	240.80
	4	285.53	275.49	279.50	271.29
7	1	629.57	604.97	604.98	585.11
	2	366.68	352.30	352.36	340.75
	3	259.87	249.71	250.85	242.72
	4	290.52	279.22	282.45	273.61

Table 7. Natural frequencies (Hz) variation versus circumferential wave numbers (n), of C–C cylindrical shells ($m=1, L/R=2, h/R=0.05$).

p	***n***	**Shell I**	**Shell II**	**Shell III**	**Shell IV**
3	1	662.27	647.28	648.26	635.13
	2	439.62	429.50	430.76	421.89
	3	336.21	328.01	330.26	323.04
	4	336.22	327.13	331.99	323.97
5	1	685.58	662.82	663.64	644.73
	2	455.03	439.84	440.95	428.31
	3	347.80	336.03	337.87	328.11
	4	347.44	335.36	339.27	329.36
7	1	648.42	671.03	671.75	649.62
	2	463.53	445.31	446.30	431.58
	3	354.19	340.30	341.84	330.70
	4	353.61	339.82	342.98	332.11

Table 8. Natural frequencies (Hz) variation versus circumferential wave numbers (n), of C–F cylindrical shells ($m=1, L/R=2, h/R=0.05$).

p	*n*	Shell I	Shell II	Shell III	Shell IV
3	1	350.96	343.10	343.70	336.20
	2	170.03	166.10	166.21	162.77
	3	147.79	143.75	146.09	142.53
	4	227.92	221.19	226.48	220.55
5	1	363.32	350.36	351.20	341.28
	2	175.98	170.11	170.12	165.25
	3	152.71	147.38	149.27	144.91
	4	235.30	226.91	231.21	224.40
7	1	370.13	355.72	355.84	343.87
	2	179.26	172.23	172.81	166.52
	3	155.41	149.35	150.89	146.13
	4	239.35	230.03	233.57	226.37

Natural frequencies (Hz) variation for three–layered cylindrical shells versus various thickness configurations

In Tables 9-11, the variation of natural frequencies (Hz) for four types of cylindrical shells is tabulated with circumferential wave numbers (n) for various thickness configurations of the middle isotropic layer. Shell parameters are taken as: $m=1$, $L/R=6$, $h/R=0.002$, $p=0.5$. Thickness of middle layer is enhanced with values: $h/6$, $h/4$, $h/2$ and $2h/3$ for circumferential wave number n=1, 2, 3 and 4. From the tables, it is apparent that as we increase the thickness of middle sandwich layer, natural frequencies for all four types of shells decrease by increasing thickness of the middle layer constructed from isotropic material. It is assumed that the ratio in which the middle layer is increased the other two inner and outer layers are decreased in the same ratio. Also, the frequency behaviour indicates that the frequencies of shell of type III and type IV are close to each other and lie between shell frequencies of type I and type II for all three end conditions namely: SS–SS, C–F and C–C. It means that the influence of edge conditions upon frequencies has not been affected by thickness configuration of middle sandwich layer. Natural frequencies of the shells for C-C boundary condition are higher than for any other boundary conditions but have lower value for the C-F boundary condition while the natural frequencies of the shells for SS-SS end condition lie between the other two end conditions.

Table 9. Natural frequencies (Hz) variation versus various thickness configurations of middle isotropic layer, for SS–SS three–layered cylindrical shells ($m = 1, L/R = 6, h/R = 0.002, p = 0.5$).

Thickness of middle layer	n	Shell I	Shell II	Shell III	Shell IV
$h/6$	1	141.72	144.46	135.73	138.01
	2	53.066	54.095	50.887	51.753
	3	25.178	25.654	24.221	24.627
	4	15.514	15.745	14.998	15.196
$h/4$	1	131.18	133.39	128.22	130.20
	2	49.870	50.715	48.746	49.508
	3	24.200	24.601	23.674	24.038
	4	15.326	15.515	14.993	15.165
$h/2$	1	129.71	130.69	126.89	127.77
	2	49.352	49.735	48.292	48.639
	3	23.970	24.154	23.480	23.649
	4	15.146	15.225	14.844	14.917
$2h/3$	1	125.07	125.59	123.37	123.85
	2	47.874	48.076	47.225	47.414
	3	23.503	23.600	23.194	23.286
	4	15.119	15.156	14.012	14.908

Table 10. Natural frequencies (Hz) variation versus various thickness configurations of middle isotropic layer, for C–C three–layered cylindrical shells ($m = 1, L/R = 6, h/R = 0.002, p = 0.5$).

Thickness of middle layer	n	Shell I	Shell II	Shell III	Shell IV
$h/6$	1	211.07	215.15	202.66	206.09
	2	96.784	98.667	93.023	94.626
	3	48.355	49.296	46.636	47.453
	4	27.414	27.921	26.559	27.008
$h/4$	1	196.41	199.72	192.35	195.33
	2	92.611	94.195	90.612	92.048
	3	48.017	48.848	47.025	47.786
	4	28.295	28.763	27.758	28.191
$h/2$	1	194.32	195.80	190.34	191.67
	2	91.761	92.499	89.871	90.543
	3	47.686	48.093	46.766	47.142
	4	28.151	28.389	27.661	27.884
$2h/3$	1	187.78	188.56	185.37	186.10
	2	89.673	90.066	88.497	88.865
	3	47.402	47.623	46.806	47.015
	4	28.592	28.723	28.256	28.382

Table 11. Natural frequencies (Hz) variation versus various thickness configurations of middle isotropic layer, for C–F three–layered cylindrical shells ($m=1, L/R=6, h/R=0.002, p=0.5$).

Thickness of middle layer	n	Shell I	Shell II	Shell III	Shell IV
$h/6$	1	61.107	62.289	58.316	59.303
	2	19.993	20.380	19.132	19.460
	3	9.5434	9.6999	9.1926	9.3264
	4	7.9705	8.0187	7.7631	7.8012
$h/4$	1	56.848	57.806	55.371	56.231
	2	19.127	19.452	18.642	18.935
	3	9.4810	9.6137	9.2614	9.3815
	4	8.2480	8.2735	8.0771	8.0971
$h/2$	1	56.200	56.627	54.857	55.244
	2	18.940	19.090	18.503	18.640
	3	9.3868	9.4468	9.1866	9.2411
	4	8.1126	8.1090	7.9512	7.9452
$2h/3$	1	54.281	54.506	53.465	53.676
	2	18.508	18.588	18.235	18.310
	3	9.3668	9.3970	9.2319	9.2601
	4	8.3123	8.3010	8.1830	8.1709

Effect of shell configurations on natural frequencies (Hz) versus circumferential wave numbers (n) with ring supports

In Tables 12-14, the behaviour of frequencies for all four types of shells are presented against the circumferential wave numbers (n) with a ring support attached at $a=0.3L$ versus power law exponents p=3, 5 and 7 under the influence of three boundary conditions: clamped–clamped (C–C), simply supported-simply supported (SS–SS) and clamped–free (C–F). The geometrical parameters of the shells are taken to be: $L/R=2$, $h/R=0.05$ and half axial wave mode is m=1. It was noted that the natural frequency increased with the increasing value of power law exponent (p) in a similar manner as observed previously in Tables 6-8 and that the natural frequencies of the shells with ring support was 30% to 40% higher as compared to the shells without ring support.

Table 12. Natural frequencies (Hz) variation versus circumferential wave numbers (*n*), for SS-SS cylindrical shells with ring support ($m=1, L/R=2, h/R=0.05, a/L=0.3$).

p	***n***	**Shell I**	**Shell II**	**Shell III**	**Shell IV**
3	1	778.65	761.67	758.82	744.02
	2	720.20	705.19	699.30	686.32
	3	728.42	714.24	705.18	693.10
	4	764.73	750.86	738.76	727.15
5	1	805.98	780.29	776.52	755.33
	2	745.34	722.86	715.18	696.85
	3	753.63	732.82	720.49	703.93
	4	790.88	771.24	753.89	738.79
7	1	821.02	790.12	785.85	761.09
	2	759.16	732.19	723.55	702.24
	3	767.46	742.64	728.54	709.49
	4	805.19	782.04	761.81	744.80

Table 13. Natural frequencies (Hz) variation versus circumferential wave numbers (*n*), for C-C cylindrical shells with ring support ($m=1, L/R=2, h/R=0.05, a/L=0.3$).

p	***n***	**Shell I**	**Shell II**	**Shell III**	**Shell IV**
3	1	794.91	777.00	776.18	760.49
	2	702.34	686.92	684.05	670.61
	3	683.06	668.87	663.43	651.21
	4	704.70	690.93	683.09	671.41
5	1	822.85	795.74	794.52	772.02
	2	726.91	703.77	699.92	680.85
	3	706.76	685.87	678.21	661.33
	4	728.83	709.28	697.47	682.12
7	1	838.23	805.63	804.19	777.88
	2	740.44	712.68	708.27	686.08
	3	719.78	694.87	685.98	666.52
	4	742.06	719.01	687.63	687.63

Table 14. Natural frequencies (Hz) variation versus circumferential wave numbers (n), for C-F cylindrical shells with ring support ($m=1, L/R=2, h/R=0.05, a/L=0.3$).

p	*n*	Shell I	Shell II	Shell III	Shell IV
3	1	428.65	418.96	417.87	409.39
	2	336.63	329.19	326.98	320.49
	3	335.71	328.55	325.92	319.75
	4	380.52	372.28	370.41	363.36
5	1	443.74	429.00	427.81	415.56
	2	348.45	337.18	334.65	325.35
	3	347.34	336.86	333.20	324.73
	4	393.41	382.09	378.21	369.26
7	1	452.06	434.29	433.06	418.70
	2	354.95	341.40	338.69	327.83
	3	353.73	341.26	337.01	327.29
	4	400.48	387.32	382.24	372.30

It is also apparent that natural frequencies for shells type III and type IV fall between the frequencies of shells type I and type II. Moreover It was noted that this shell frequency spectra contrasted with shell frequency behaviour for a cylindrical shell without a ring support, where shell frequencies of type II and type IV lie between the frequencies of shells type I and type III. Furthermore, it is evident from Tables 12-14, that for shells frequencies with clamped–free end condition, the natural frequencies are the smallest ones among those of clamped–clamped and simply supported–simply supported edge conditions.

Natural frequencies (Hz) variation for three-layered cylindrical shells with ring supports against various thickness configurations

In Fig. 4, variation of natural frequencies of three–layered cylindrical shells with a ring support are sketched against the circumferential wave numbers, n, for three end conditions: SS–SS, C–C and C–F, respectively, for different patterns of thickness configurations. The geometrical parameters for cylindrical shell are: $L/R=6$, $h/R=0.002$ and $p=0.5$. The circumferential wave number n is assumed to vary from 1 to 10. The ring support is attached at $a/L=0.1$. The thickness of the middle layer is increased with values $h/6, h/4, h/2$ and $2h/3$, for axial half wave number m=1. As the thickness of middle sandwich layer is enhanced, shell frequency diminishes. The influence of variation of shell layer's thickness on its natural frequencies is significant. The frequency trend of all three boundary conditions is found to be similar, as the lowest frequency corresponds to the fundamental circumferential mode ($n=1$) for all three boundary conditions.

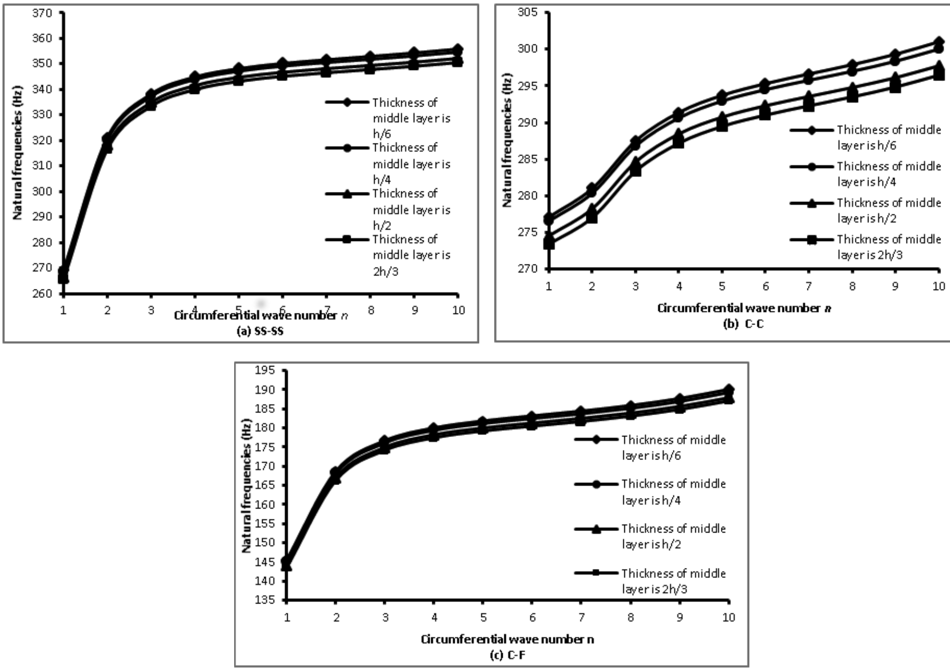

Fig. 4. Natural frequencies (Hz) variation versus various thickness configurations of middle isotropic layer, for three – layered a (a) SS – SS, (b) C-C, (c) C-F cylindrical shell of Type I ($m=1, L/R=6, h/R=0.002, a/L=0.1, p=0.5$).

It is also observed that, as *n* is increased, shells frequencies increase gradually but the rate of increment of the shells frequencies is higher initially from *n=1* to *n=3*, This rate of increment slows down for higher values of n > 3 whereas in case of C-C boundary condition, rate of increment of the natural frequencies from *n=2* to *n=3* is highest. Moreover curves of the frequencies for different configurations of layer thickness are seen neither to be converging nor diverging. Natural frequencies of the SS-SS and C-F cylindrical shells have similar behaviour and for higher values of circumferential wave number *n,* natural frequencies of the shells for different patterns of the thickness of layers, seem to diverge. However, for lower value of *n,* natural frequencies seem to have converging behaviour.

Natural frequencies (Hz) variation for three – layered cylindrical shells versus a/L

In Fig.5, variation of the natural frequencies (Hz) of four types of three–layered cylindrical shells, for various L/R ratios is plotted against different positions of a ring support under the influence of three edge conditions: SS–SS, C–C and C-F. The shell geometrical parameters are: $m=1$, $n=1$, $L/R=10, 15, 20$, $h/R=0.002$ and $p=0.5$.

The thickness of each layer of the shells is assumed to be $h/3$. It is observed that the natural frequencies of the shells have been affected by the location of ring support in the shell and this effect varies with end conditions. It is evident from these figures that the natural frequencies increase with the increasing values of *a/L*, (*0 < a/L < 0.5)* and at *a/L=0.5* it attains its highest values while for *0.5 < a/L < 1* it begins to decrease with the increasing value of *a/L* ratio and mold itself as bell shape symmetric curve but only for the case of SS-SS and C-C symmetric boundary conditions. It is clearly seen that natural frequency acquires its greatest value for all four types of cylindrical shells, when the ring support is placed at centre of the cylindrical shell, for all L/R ratios.

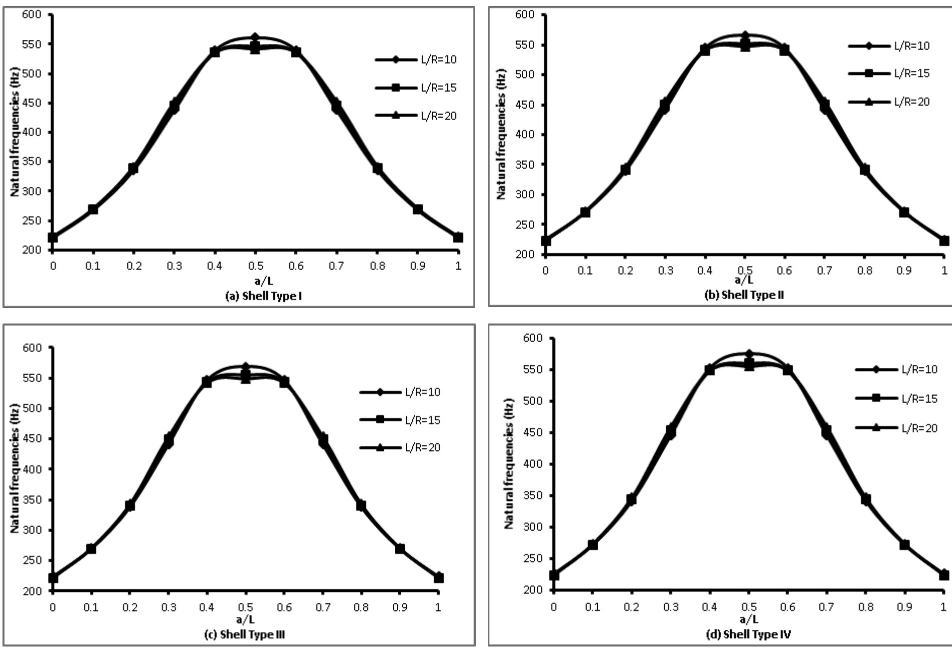

Fig. 5. Natural frequencies (Hz) variation with position of the ring support for various L/R ratios, of (a) Type I, (b) Type II, (c) Type III and (d) Type IV shells for SS–SS boundary condition ($m = 1, n = 1, h/R = 0.002$, $p = 0.5$).

It is observed that as ring support displaces from centre towards either side of cylindrical shell, the natural frequencies decrease. Thus, the natural frequency curve is symmetrical about centre of a cylindrical shell for symmetric end conditions enforced on both shell ends. Also for SS–SS cylindrical shells of type I and type II, the natural frequency difference between *a/L*=0 and 0.5 with *L/R*=10 is 60.3%, with *L/R*=15 is 59.6% and for *L/R* =20 is found to be 59.2%. For SS–SS cylindrical shells of type III and type IV, the difference between natural frequencies for *a/L*=0 and 0.5 with *L/R*=10 is 60.7%, for *L/R* =15 is 60% and for *L/R* =20 is 59.5%. In Fig. 6, for C–C cylindrical shells of type I and type II, the difference between natural frequencies for *a/L*=0 and 0.5 at *L/R*=10, 15 and 20 is 71.4%, 70% and 68.6%, respectively, and for

C–C cylindrical shells of type III and type IV, this difference is 71.5%, 70.2% and 68.9% for *L/R* =10, 15 and 20, respectively. For clamped–free (C–F) end condition for four type cylindrical shells, natural frequency curve is not symmetrical about centre due to the different boundary conditions at both ends as shown in Figure 7. It is also inferred, that frequencies related to clamped–free end condition are the smallest as compared to those for the clamped–clamped and simply supported–simply supported edge conditions. Moreover, It is noted that the four type of shells attain their peak values at ring support-to-length ratio *a/L* from 0.7 to 0.9 for C-F boundary condition, from 0.4 to 0.6 for SS–SS shell and for C–C shells. However, it is also observed that for short shell the natural frequencies attain bell shape due to symmetric boundary condition at both ends of the shells. Furthermore, it is noted that natural frequencies of short shell is higher than that of long shell for all types of shells and for the three types of boundary conditions. The influence of ring support is seen to be very prominent in the variation of natural frequencies of all types of cylindrical shells with three types of selected boundary conditions.

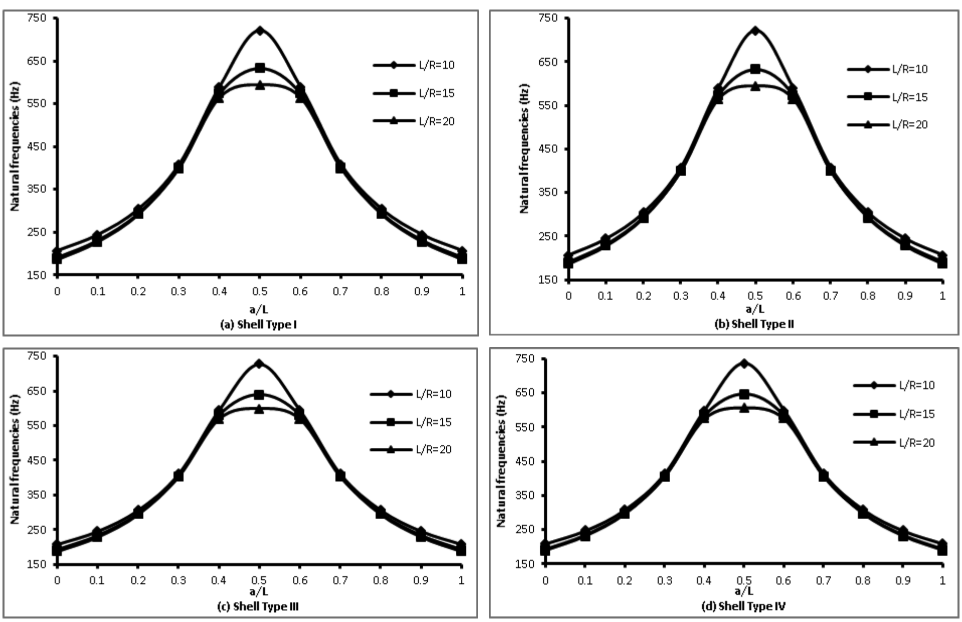

Fig. 6. Natural frequencies (Hz) variation with position of the ring support for various L/R ratios, for C–C boundary condition of (a) Type I, (b) Type II, (c) Type III and (d) Type IV shells ($m=1, n=1, h/R=0.002$, $p=0.5$).

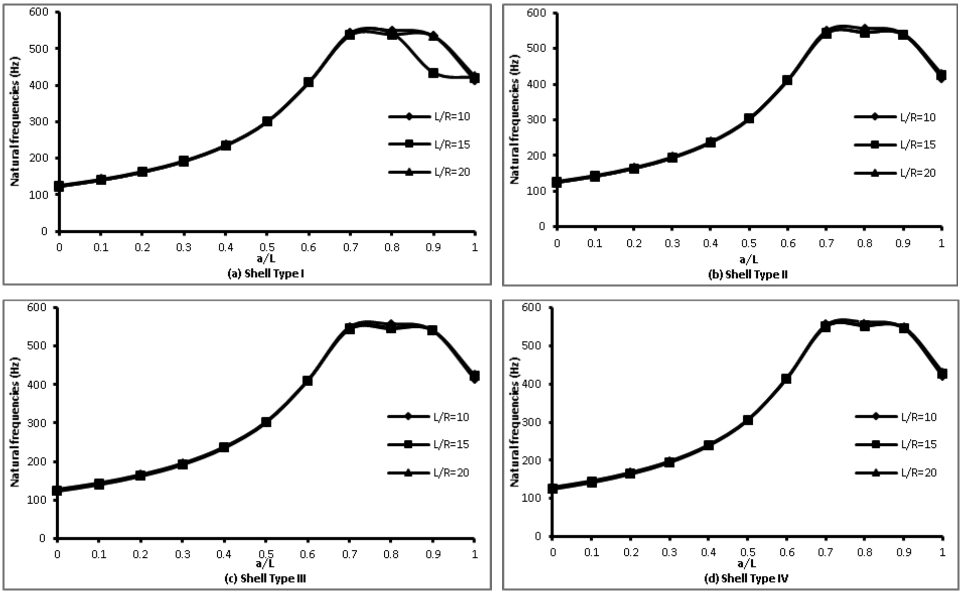

Fig. 7. Natural frequencies (Hz) variation with position of the ring support for various L/R ratios, for C – F boundary condition of (a) Type I, (b) Type II, (c) Type III and (d) Type IV shells ($m=1, n=1, h/R=0.002$, $p=0.5$).

CONCLUSION AND RECOMMENDATIONS

This work presents the influence of ring supports on free vibrations of functionally graded three layered sandwich cylindrical shells with middle sandwich layer consisting of isotropic material whereas the inner and outer layers are of FGMs. By changing the order of functionally graded constituent materials and by keeping the same material for the isotropic middle layer, four types of shells are constructed and their frequencies variation are analyzed for three end conditions namely: C–C, SS–SS and C–F. It is observed that the effect of power law exponent is found to be similar for all four types of shells with and without ring supports, since the frequency increases as we increase the power law exponent. By increasing the thickness of the middle sandwich layer, the natural frequency decreases for all four types of shells with and without ring support for all three boundary conditions. The induction of ring support on cylindrical shells has prominent effect on the natural frequency as compared to the shell frequency without a ring support. Minimum frequency is related to fundamental circumferential mode ($n=1$) for all thickness configurations of the middle layer for all types of shells. The variation of the natural frequency of the shells with location of ring support with various L/R ratios is also examined. It is concluded that this frequency curve is symmetrical about centre of the cylindrical shell for C–C and SS–SS end conditions and is not symmetrical for C–F boundary condition. Frequency response for shells type I and type II is the same whereas, for shells of type III and type IV, the frequency behaviour is found to be similar. This work can be extended to examine other vibration behaviour like buckling, post buckling, mode shapes of ring–stiffened multi-layered cylindrical shells with the varying thickness of the layers.

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APPENDIX-I:

$$a_{11} = A_{11}I_1 + \frac{n^2 A_{66}}{R^2} I_2$$

$$a_{12} = \frac{nA_{12}}{R} I_3 - \frac{nA_{66}}{R} I_2 + \frac{nB_{12}}{R^2} I_3 - \frac{2nB_{66}}{R^2} I_2$$

$$a_{13} = \frac{A_{12}}{R} I_4 - B_{11}I_5 - 2B_{11}I_6 + \frac{n^2 B_{12}}{R^2} I_4 - \frac{2n^2 B_{66}}{R^2} I_7 - \frac{2n^2 B_{66}}{R^2} I_8$$

$$a_{22} = \frac{n^2 A_{22}}{R^2} I_9 + A_{66} I_2 + \frac{2n^2 B_{22}}{R^3} I_9 + \frac{4B_{66}}{R} I_2 + \frac{n^2 D_{22}}{R^4} I_9 + \frac{4D_{66}}{R^2} I_2$$

$$a_{23} = \frac{nA_{22}}{R^2} I_{10} - \frac{nB_{12}}{R} I_4 - \frac{2nB_{12}}{R} I_8 + \frac{n^3 B_{22}}{R^3} I_{10} + \frac{nB_{22}}{R^3} I_{10} + \frac{2nB_{66}}{R} I_7 + \frac{2nB_{66}}{R} I_8 + \frac{n^3 D_{22}}{R^4} I_{10}$$
$$- \frac{nD_{12}}{R^2} I_4 - \frac{2nD_{12}}{R^2} I_8 + \frac{4nD_{66}}{R^2} I_7 + \frac{4nD_{66}}{R^2} I_8$$

$$a_{33} = \frac{A_{22}}{R^2} I_{11} - \frac{2B_{12}}{R} I_{12} - \frac{4B_{12}}{R} I_{13} + \frac{2n^2 B_{22}}{R^3} I_{11} + D_{11} I_{14} + 4D_{11} I_2 + 4D_{11} I_{15} + \frac{n^4 D_{22}}{R^4} I_{11} - \frac{2n^2 D_{12}}{R^2} I_{12}$$
$$- \frac{4n^2 D_{12}}{R^2} I_{13} + \frac{4n^2 D_{66}}{R^2} I_{16} + \frac{4n^2 D_{66}}{R^2} I_9 + \frac{8n^2 D_{66}}{R^2} I_{13}$$

where

$$I_1 = \int_0^L \left(\frac{d^2\varphi}{dx^2}\right)^2 dx, \qquad I_2 = \int_0^L \left(\frac{d\varphi}{dx}\right)^2 dx, \qquad I_3 = \int_0^L \varphi \frac{d^2\varphi}{dx^2} dx$$

$$I_4 = \int_0^L (x-a)\varphi \frac{d^2\varphi}{dx^2} dx, \qquad I_5 = \int_0^L (x-a)\left(\frac{d^2\varphi}{dx^2}\right)^2 dx, \qquad I_6 = \int_0^L \left(\frac{d\varphi}{dx}\right)\left(\frac{d^2\varphi}{dx^2}\right) dx$$

$$I_7 = \int_0^L (x-a)\left(\frac{d\varphi}{dx}\right)^2 dx, \qquad I_8 = \int_0^L \varphi \frac{d\varphi}{dx} dx, \qquad I_9 = \int_0^L \varphi^2 dx$$

$$I_{10} = \int_0^L (x-a)\varphi^2 dx, \qquad I_{11} = \int_0^L (x-a)^2 \varphi^2 dx, \qquad I_{12} = \int_0^L (x-a)^2 \varphi \frac{d^2\varphi}{dx^2} dx$$

$$I_{13} = \int_0^L (x-a)\varphi \frac{d\varphi}{dx} dx, \qquad I_{14} = \int_0^L (x-a)^2 \left(\frac{d^2\varphi}{dx^2}\right)^2 dx, \qquad I_{15} = \int_0^L (x-a) \frac{d\varphi}{dx}\frac{d^2\varphi}{dx^2} dx,$$

$$I_{16} = \int_0^L (x-a)^2 \left(\frac{d\varphi}{dx}\right)^2 dx$$