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Forced vibration analysis of inhomogeneous rods with non-uniform cross-section

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ABSTRACT

Forced vibration analysis of cantilever rods is presented that have material properties and cross-section areas that arbitrarily vary in the axial direction, solved using Laplace transform in time domain and complementary functions method (CFM) in the spatial domain. Under the Laplace transformation, the partial differential equation is transformed into time-independent boundary value problem in the axial direction, which is solved by CFM. Then, inverse transform is taken by modified Durbin's method into the time domain. In the end, the non-dimensional displacement results are compared with both benchmark and finite element method (FEM) solutions available in the literature. In addition to satisfying a fair amount of accuracy with small computational costs, the approach presented in this study is well-structured, simple, and efficient.

Keywords: Inhomogeneous; non-uniform; complementary functions method; Laplace transform.

INTRODUCTION

Current engineering designs frequently involve situations where inhomogeneous materials are used intentionally to attain the required structural performance. The analysis of inhomogeneous structural members (rods, beams, shafts, pipes, tubes, etc.) is quite important, especially in engineering design. The potential uses of these members in engineering applications include aerospace structures, engine parts, fusion energy devices, and other engineering structures. The use of inhomogeneous members can help the designer reduce the weight and stress intensity factors; improve residual stress distribution, high temperature withstanding ability, and strength and stability of structures. Their material properties can be designed as to improve the behaviour of structures in which they are embedded. As the amount of these structures in different application increases, different techniques and methodologies need to be employed in terms of characterization to design and analyse the structural components made of inhomogeneous members.

Different studies have been carried out by researchers on the static and dynamic behaviour of inhomogeneous structural members such as beams (Tong *et al*., 1995; Elishakof & Candan, 2001; Avcar, 2015-2016; Yang et al., 2008; Shahba *et al*., 2011; Huang & Luo, 2011; Huang *et al*., 2013), annular circular plates or disks (Horgan, 1999; Efraim & Eisenberger, 2007), axial bars (Maalawi, 2011; Celebi *et al*., 2012; Akgoz & Civalek, 2013; Hong *et al*., 2014; Hong & Lee, 2015), shells (Mecitoglu, 1996), and spheres (Heyliger & Jilani, 1992). Inhomogeneity in rods can arise due to variation in cross-sectional area or in density (and thus Young's modulus). Abrate (1995) found exact solutions for rods and beams with polynomial cross-sections and inertia. Kumar *et al*. (1997) studied the longitudinal displacement of several rods with variable cross-sections and figured out the analytical solutions of the longitudinal displacements. Li (2000) carried out a functional transformation of the governing differential equation and then obtained exact solutions for certain functional forms of an involved parameter. Yardimoglu and Aydin (2011) obtained the longitudinal natural vibration frequencies of rods with variable cross-sections by using the transformation method. Celebi *et al*. (2011) found exact solutions for forced vibration of non-uniform rods by using Laplace transformation method. Shokrollahi *et al*. (2014) studied the non-dimensional natural frequencies of rods with various area cross-sections by using discrete singular convolution method. Conway *et al*. (1964) obtained an exact solution for a conical beam in terms of Bessel functions. Candan and Elishakoff (2001) constructed several closed-form solutions for inhomogeneous rods with continuously variable moduli of elasticity. Elishakoff and Perez (2006) obtained closed form solution for the free vibration of inhomogeneous bar with a tip mass. Nachum and Altus (2007) studied natural frequencies and mode shapes of nonhomogeneous rods and beams based on the functional perturbation method. Calio and Elishakoff (2008) investigated a special class of closed-form solutions for inhomogeneous rod. Celebi *et al*. (2012) analysed the axial vibration of inhomogeneous rod modelled as a continuous system. Static and dynamics analyses of the inhomogeneous structures are obtained via different solution methods, including analytical methods (Menaa *et al*., 2012; Celebi & Tutuncu, 2014; Huang *et al*., 2013), modal analysis method (Murin *et al*., 2010 -2013), power series expansion methods (Calim, 2009; Horgan, 2007; Maalawi, 2011), differential quadrature method (DQM) (Xiang & Yang, 2008), dynamic stiffness method (Efraim & Eisenberger, 2007), numerical methods such as finite element method (FEM) (Chakraborty *et al*., 2003; Piovan & Sampaio, 2008; Alshorbagy *et al*., 2011; Shahba *et al*., 2011), and complementary functions method (Tutuncu & Temel, 2009; Celebi *et al*., 2016 -2017).

The objective of this paper is to present a unified approach for analysing forced vibration of inhomogeneous rods. The material properties and cross-sectional areas of the rods are varying arbitrarily in the axial direction, and time-dependent forcing functions are applied at the free end of the rod. The inhomogeneity properties including variable sectional area, modulus of elasticity, and mass density often produce an irregular and variable coefficient governing differential equation. Analytical solutions of such equations may be possible by superposition methods with combination of power series methods, but it is not practical and also very specific to elementary geometries, material properties, and rather particular types of loadings. Therefore, at this stage, numerical solution is becoming essential. In the numeric analysis of the present study, Laplace transform and complementary functions method (CFM) are combined in order to solve the partial differential equations with irregular variable coefficients. Under the Laplace transformation, the partial differential equation is transformed into time-independent boundary-value problem, which can be solved by any standard method in the literature. The system of the initial-value problem is solved by the fifth-order Runge-Kutta method. Inverse transformation of the results into the time domain is taken by modified Durbin's method. The theoretical background for the method is available in the literature (Tutuncu & Temel, 2009; Celebi *et al*., 2016-2017). A detailed form of solution is first offered for an inhomogeneous rod with uniform cross-section and compared to reference data that is available in the literature (Celebi *et al*., 2012) to validate the numerical results. After validating the proposed approach, the analyses of inhomogeneous rods with non-uniform cross-section are studied in detail. The efficiency of the present method is demonstrated by comparing the results with Newmark integration method. Specifically, the new attributes of this method are as follows: (1) the forced vibration response is directly obtained; (2) performing forced vibration analysis and determination of natural frequencies and mode shapes are not needed; (3) the solution procedure can be applied to any choice of material and cross-sectional area model. The solution procedure is well structured, simple, and efficient.

GOVERNING EQUATION

Figure 1 shows the geometry of the inhomogeneous rod with the non-uniform cross-section considered in this study.

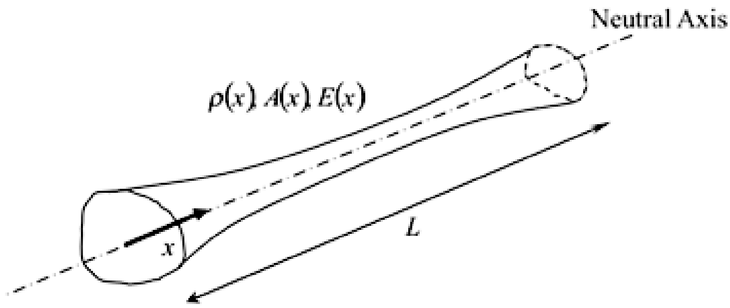


Figure 1. Inhomogeneous rod with non-uniform cross-section.

The partial differential equation that governs the axial dynamic behaviour of a vibrating rod is (Clough & Penzien, 1993)

$$\frac{\partial}{\partial x}\left[E(x)\,A(x)\frac{\partial u(x,t)}{\partial x}\right]-\rho(x)\,A(x)\frac{\partial u^2(x,t)}{\partial t^2}=p(x,t) \tag{1}$$

Usually, the external axial loading consists only of end loads, in which case the right hand side of the Eq. (1) would be zero. However, when solving Eq. (1), the boundary conditions imposed at $x = 0$ and $x = L$ must be satisfied. Eq. (1) can be written as

$$\frac{1}{E(x)}\frac{\partial E(x)}{\partial x}\frac{\partial u(x,t)}{\partial x}+\frac{1}{A(x)}\frac{\partial A(x)}{\partial x}\frac{\partial u(x,t)}{\partial x}+\frac{\partial u^2(x,t)}{\partial x^2}=\frac{\rho(x)}{E(x)}\frac{\partial u^2(x,t)}{\partial t^2} \tag{2}$$

where (x) denotes the axial coordinate, $u(x,t)$ axial displacement at any position x and time t, $E(x)$ modulus of elasticity, $p(x)$ mass density, and $A(x)$ cross-sectional area.

By using the dimensionless variables

$$v(\eta,\tau)=\frac{u(x,t)}{L},\eta=\frac{x}{L},\tau=\frac{c\,t}{L} \tag{3}$$

renders Eq. (2) in the form

$$\frac{1}{E(\eta)}\frac{\partial E(\eta)}{\partial \eta}\frac{\partial v(\eta,\tau)}{\partial \eta}+\frac{1}{A(\eta)}\frac{\partial A(\eta)}{\partial \eta}\frac{\partial v(\eta,\tau)}{\partial \eta}+\frac{\partial v^2(\eta,\tau)}{\partial \eta^2}=c^2\frac{\rho(\eta)}{E(\eta)}\frac{\partial v^2(\eta,\tau)}{\partial \tau^2} \tag{4}$$

where $c^2 = E_0/\rho_0$ is the speed of longitudinal waves.

The rod is fixed at $\eta = 0$. At the right end $\eta = 1.0$, the dynamic axial force applied to the rod, the initial and boundary conditions are

$$v(\eta,0)=0\,,\qquad \frac{\partial v(\eta,0)}{\partial \tau}=0 \tag{5}$$

$$v(0,\tau)=0\,,\qquad \frac{\partial v(1,\tau)}{\partial \eta}=\frac{P(\tau)}{A(1)E(1)} \tag{6}$$

DEFINITION OF THE ROD MODEL

To ascertain the effect of the inhomogeneity, two different material models are considered with the following physical parameter distribution: (a) power law material model- $E(\eta) = E_0(1 + a\eta)^2$, $\rho(\eta) = \rho_0(1 + a\eta)^2$ and (b) exponential law material model- $E(\eta) = E_0 e^{-a\eta}$, $\rho(\eta) = \rho_0 e^{-a\eta}$. In addition to this, the cross-section is assumed to vary along the non-dimensional axial coordinate η in the forms $A(\eta) = A_0 sin^2[a\eta + b]$, $A(\eta) = A_0(1 + a\eta)^2$, $A(\eta) = A_0 e^{-a\eta}$. Here, E_0, ρ_0, A_0, b are real constants and (a) is an inhomogeneity parameter.

In this study, both geometric and physical parameters are varying together. Differential equation (4) will be solved for several cases as follows:

Case 1. $E(\eta) = E_0(1 + a\eta)^2$ - $\rho(\eta) = \rho_0(1 + a\eta)^2$, $A(\eta) = A_0 sin^2[a\eta + b]$

Case 2. $E(\eta) = E_0 e^{-a\eta}$ - $\rho(\eta) = \rho_0 e^{-a\eta}$, $A(\eta) = A_0(1 + a\eta)^2$

Case 3. $E(\eta) = E_0(1 + a\eta)^2$ - $\rho(\eta) = \rho_0(1 + a\eta)^2$, $A(\eta) = A_0 e^{-a\eta}$

Case 4. $E(\eta) = E_0 e^{-a\eta}$ - $\rho(\eta) = \rho_0 e^{-a\eta}$, $A(\eta) = A_0 sin^2[a\eta + b]$

Case 5. $E(\eta) = E_0(1 + a\eta)^2$ - $\rho(\eta) = \rho_0(1 + a\eta)^2$, $A(\eta) = A_0$

Case 6. $E(\eta) = E_0 e^{-a\eta}$ - $\rho(\eta) = \rho_0 e^{-a\eta}$, $A(\eta) = A_0$

Analytical solution can only be obtained for case 5-6 among the above cases; the other cases have no analytical results. A detailed analysis for one of the cases is presented below, and subsequently, the results for other cases, which have the same solution method, are provided.

SOLUTION FOR SINUSOIDAL CROSS-SECTION ROD WITH POLYNOMIAL VARYING DENSITY AND MODULUS OF ELASTICITY

This case presents an example of an inhomogeneous rod with non-uniform cross-section in which all geometric and physical parameters vary with non-dimensional parameter η as $A(\eta) = A_0 sin^2[a\eta + b]$, $E(\eta) = E_0(1 + a\eta)^2$, $\rho(\eta) = \rho_0(1 + a\eta)^2$. Substituting chosen parameters into Eqs. (4-6), right after taking the Laplace transform, yields

$$y^{II}(\eta, s) + \left(2a\cot[a\eta + b] + \frac{2a}{(1 + a\eta)}\right) y^{I}(\eta, s) - s^2\, y(\eta, s) = 0 \tag{7}$$

$$y(0, s) = 0\,, \qquad \frac{\partial y(1, s)}{\partial \eta} = \frac{L\{P(\tau)\}}{A_0 \sin^2[a + b]\, E_0 (1 + a)^2} \tag{8}$$

homogeneous boundary value problem, where $y(\eta, p) = L\{v(\eta, \tau\}$, with s being the complex Laplace parameter.

At this stage, a general closed-form solution of Eq. (7) is not practical. So, solutions of Eq. (7) in the Laplace domain are carried out by CFM, which is based on the transformation of the solutions of second-order boundary value problems to a system of initial value problems. The solution of Eq. (7) is

$$y = b_i y_i \quad , \quad i = 1{,}2 \tag{9}$$

where y_i is the linearly independent homogeneous solution, b_i is the constant to be determined with boundary conditions (8).

Letting $y_i = Z_1^{(i)}$ and $y_i^{I} = Z_2^{(i)}$ yields

$$\left(Z_1^{(i)}\right)^{I} = Z_2^{(i)}$$

$$\left(Z_2^{(i)}\right)^{I} = -(2a\cot[a\eta + b] + \frac{2a}{(1 + a\eta})Z_2^{(i)} + s^2 Z_1^{(i)} \tag{10}$$

The above system of equations could be solved numerically by CFM. The unreal initial conditions for this system of equations are selected arbitrarily, but linear independent, in order to guarantee the linear independence of the solution

$$Z_j^{(i)} = C_{ij} \quad , \quad i = 1{,}2 \tag{11}$$

where C_{ij} are arbitrary constants. This system of initial value problem is solved by the fifth-order Runge-Kutta method (RK5). The solutions are performed at only 21 collocation points through the length of 20 intervals in the interval $0 \leq \eta \leq 1$.

After having determined the homogeneous solutions y_i and their derivative y_i^{I} , by imposing these solutions into Eq. (8), one can now find two equations for constants b_i. These equations, in a matrix form, can be written as

$$\begin{bmatrix} C_{11} & C_{21} \\ y_1^{I}(1, s) & y_2^{I}(1, s) \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 0 \\ \dfrac{L\{P(\tau)\}}{A_0 \sin^2[a + b]\, E_0 (1 + a)^2} \end{bmatrix} \tag{12}$$

Here, C_{11} and C_{21} correspond to $y_1(0, s)$ and $y_2(0, s)$, respectively.

The non-dimensional displacement values $v(\eta, \tau)$ in the time domain can be obtained numerically from the inverse Laplace transform. For this purpose, the modified Durbin's inverse Laplace transform technique based on fast Fourier transform (FFT) is used. Details of the technique are given in Appendix A.

RESULTS

In this study, a general objective computer program is coded in MATHEMATICA to analyse the forced vibration of inhomogeneous rods with non-uniform cross-section (Abell & Braselton, 2004). In the solution procedure of the initial value problem based on CFM, fifth-order Runge-Kutta algorithm is used. Inverse transformation into the time domain is taken by the modified Durbin's method. Three types of dynamic axial end force will be used in the analysis: $P_1(\tau) = P_0(1 - cos[\gamma\tau])$, $P_2(\tau) = P_0$ and $P_3(\tau) = P_0(1 - e^{-\gamma\tau})$ for γ=0.6. The material and geometrical models for different cases are given in the definition of the rod section. The inhomogeneity parameter "*a*" is taken as 0, 1, 2 for all cases considered. The results for "*a=0*" correspond to the uniform cross-section with constant material properties.

The efficacy and adequacy of the present method are first compared to the analytical results presented for inhomogeneous rods with uniform cross-section (Celebi *et al*., 2012). The comparison will be illustrated in Tables 1–6. It can be noted from the tables that the CFM results are complementary to the analytical solutions in Celebi *et al*. (2012) than ANSYS results. Examining these results reveals the great accuracy and efficiency achieved by CFM; calculations performed at only 21 points through the length yielded exact numerical results.

Table 1. Comparison of CFM results with analytical and numerical results for case 5 under $P_1(\tau)$.

	$v(1,\tau)$								
	a=0			a=1			a=2		
τ	CFM	Celebi *et al.*, 2012	ANSYS	CFM	Celebi *et al.*, 2012	ANSYS	CFM	Celebi *et al.*, 2012	ANSYS
5	2.13338	2.133376	2.12915	1.30411	1.304106	1.30111	0.881667	0.881667	0.879087
10	-0.23388	-0.23388	-0.23452	-0.0371	-0.0371	-0.0373	-0.36271	-0.36271	-0.36140
20	0.179785	0.179786	0.176996	-0.09914	-0.09914	-0.1002	0.059314	0.059314	0.057668
30	0.109509	0.109508	0.108413	-0.08487	-0.08487	-0.08609	-0.15767	-0.15767	-0.15662
40	0.659848	0.659849	0.654500	0.07812	0.07812	0.077231	0.216753	0.216752	0.213320
50	0.688	0.688	0.686527	0.374068	0.374067	0.372233	0.190823	0.190822	0.191549

Table 2. Comparison of CFM results with analytical and numerical results for case 5 under $P_2(\tau)$.

	$v(1,\tau)$								
	a=0			a=1			a=2		
τ	CFM	Celebi *et al.*, 2012	ANSYS	CFM	Celebi *et al.*, 2012	ANSYS	CFM	Celebi *et al.*, 2012	ANSYS
5	0.999845	1.000407	0.982324	0.107887	0.107655	0.112122	0.301791	0.301887	0.306523
10	1.97439	1.97858	1.90338	0.252835	0.253724	0.236228	0.64419	0.644694	0.632406
20	0.030772	0.0264007	0.115879	0.616221	0.615039	0.647566	0.068131	0.067815	0.071693
30	1.97673	1.97858	1.87385	0.892067	0.893377	0.896216	0.557082	0.557428	0.565088
40	0.032116	0.026401	0.150149	0.910138	0.908685	0.882136	0.159587	0.159312	0.150591
50	1.99163	1.978836	1.82829	0.578499	0.578521	0.585328	0.437142	0.43645	0.434744

Table 3. Comparison of CFM results with analytical and numerical results for case 5 under $P_3(\tau)$.

	$v(1,\tau)$								
	a=0			a=1			a=2		
τ	CFM	Celebi *et al.*, 2012	ANSYS	CFM	Celebi *et al.*, 2012	ANSYS	CFM	Celebi *et al.*, 2012	ANSYS
5	0.696886	0.696882	0.694604	0.479316	0.479316	0.477698	0.450853	0.450851	0.448833
10	1.10445	1.104459	1.09737	0.584101	0.584096	0.582325	0.450136	0.450135	0.448745
20	0.898295	0.898296	0.903889	0.705466	0.705475	0.702761	0.192374	0.192377	0.190641
30	1.1067	1.106679	1.09402	0.659433	0.659431	0.659555	0.493461	0.493458	0.492180
40	0.898239	0.898302	0.909387	0.494009	0.494003	0.490425	0.169128	0.169133	0.169414
50	1.10715	1.106679	1.08915	0.337535	0.337593	0.335158	0.493381	0.493355	0.489925

Table 4. Comparison of CFM results with analytical and numerical results for case 6 under $P_1(\tau)$.

	$v(1,\tau)$								
	a=0			a=1			a=2		
τ	CFM	Celebi *et al.*, 2012	ANSYS	CFM	Celebi *et al.*, 2012	ANSYS	CFM	Celebi *et al.*, 2012	ANSYS
5	2.13294	2.133376	2.12915	3.43205	3.432317	3.42647	6.57568	6.577714	6.57017
10	-0.23283	-0.23388	-0.23456	0.067883	0.06748	0.065364	-0.12812	-0.13023	-0.13670
20	0.179831	0.179786	0.176996	0.276992	0.276589	0.275055	0.425383	0.424095	0.422169
30	0.1105	0.109508	0.108413	0.613708	0.613464	0.610443	1.04217	1.04133	1.02593
40	0.659759	0.659849	0.654500	1.03681	1.036546	1.03251	1.67432	1.672538	1.66657
50	0.688788	0.688	0.686527	1.51709	1.516876	1.51444	2.84346	2.843814	2.82331

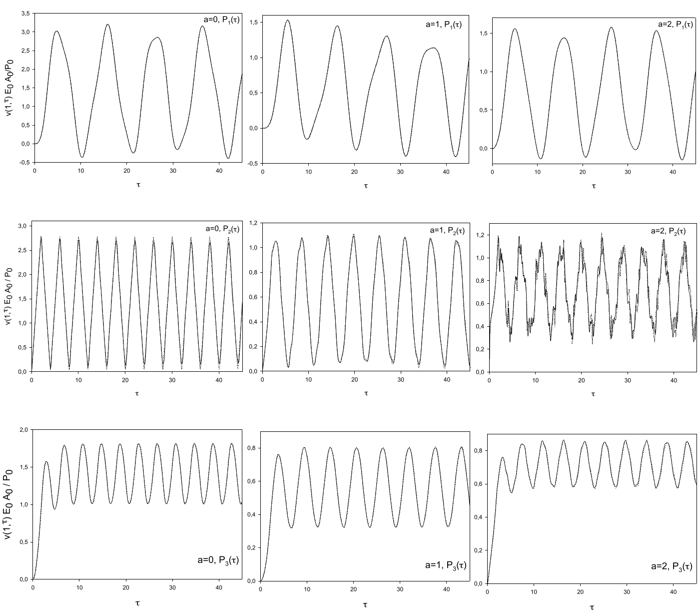
Table 5. Comparison of CFM results with analytical and numerical results for case 6 under $P_2(\tau)$.

	$v(1,\tau)$								
	a=0			a=1			a=2		
τ	CFM	Celebi *et al.*, 2012	ANSYS	CFM	Celebi *et al.*, 2012	ANSYS	CFM	Celebi *et al.*, 2012	ANSYS
5	1.00014	1.000407	0.982324	2.82698	2.829704	2.85501	2.20688	2.193008	2.30300
10	1.95421	1.978580	1.90338	0.706242	0.750206	0.682098	4.27086	4.344166	4.46342
20	0.0508049	0.0264	0.115879	0.726567	0.691595	0.608808	2.37251	2.357593	2.07136
30	1.95421	1.97858	1.87385	0.485277	0.496762	0.481202	2.14177	2.093442	2.31998
40	0.050796	0.026401	0.150149	0.824117	0.810774	0.733318	4.24035	4.276586	4.52993
50	1.95623	1.978836	1.82829	1.1062	1.102023	1.14669	0.75352	0.660371	1.42154

Table 6. Comparison of CFM results with analytical and numerical results for case 6 under $P_3(\tau)$.

$v(1,\tau)$									
	a=0			a=1			a=2		
τ	CFM	Celebi *et al.*, 2012	ANSYS	CFM	Celebi *et al.*, 2012	ANSYS	CFM	Celebi *et al.*, 2012	ANSYS
5	0.697456	0.696882	0.694604	1.79934	1.800494	1.79658	3.45606	3.458083	3.45855
10	1.10394	1.104459	1.09737	1.57887	1.57961	1.57023	3.5996	3.606126	3.59279
20	0.89885	0.898296	0.903889	1.47445	1.471508	1.47719	2.72247	2.717392	2.72499
30	1.10616	1.106679	1.09402	1.43238	1.431784	1.44200	3.62238	3.62549	3.65290
40	0.898855	0.898302	0.909387	1.42607	1.424038	1.43205	3.03772	3.036661	3.05766
50	1.10617	1.106679	1.10700	1.38319	1.3824	1.37667	3.10921	3.108574	3.16154

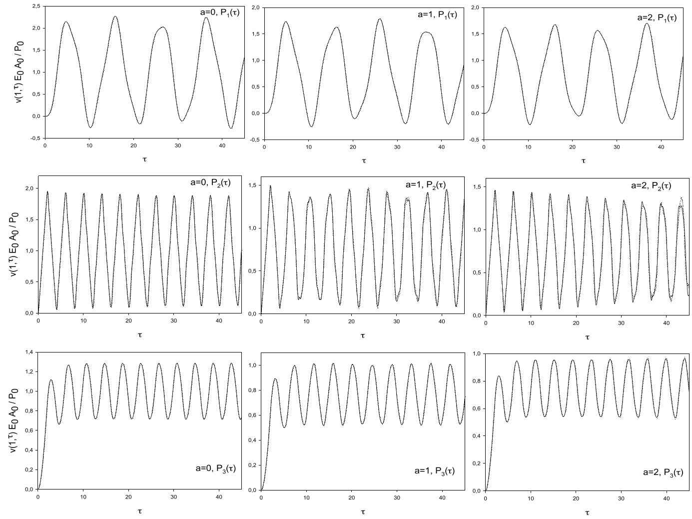
Having testing the validity of the present method, the forced vibration analysis of the inhomogeneous rod with non-uniform cross-section is now presented. Figures 2–5 show the non-dimensional end displacements obtained using CFM.



······*CFM* , —— *ANSYS*

Figure 2. Non-dimensional end displacements for case 1 under dynamic loadings.

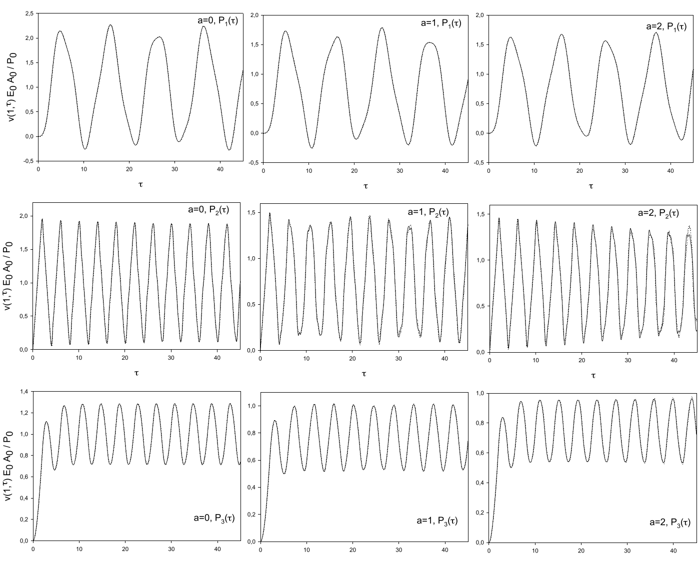
The numerical CFM results are compared with the results obtained by a commercial FEM package, ANSYS. The Newmark integration method was preferred in the analysis to solve the transient load problem. The element type, Beam-188, is used to compute the non-dimensional end displacements with ANSYS. Element section properties and material properties were changed with the axial direction by use of ANSYS parametric design language (APDL). The same geometrical and physical parameters given in the definition of the bar section are applied for computation. The FEM model of ANSYS has 1000 elements of equal length in the axial direction. A good agreement between the results obtained using CFM and those given by the FEM has been observed from the figures.



……*CFM* , —— *ANSYS*

Figure 3. Non-dimensional end displacements for case 2 under dynamic loadings.

As shown in Fig. 2–4 , the displacement amplitude decreases with increasing inhomogeneity parameter *"a"* for case 1, case 2, and case 3, whereas for case 4, the displacement amplitude increases with increasing inhomogeneity parameter . As can be seen from Fig. 5, the inhomogeneity parameter is more effective for case 4. The greater displacement amplitude differences between inhomogeneity parameters are obtained for this case. The oscillations of case 2 and case 3 are observed to be similar for all loading types. According to this, we can say that the interchange of the expressions defining the material and geometrical properties given in cases yields the same result.

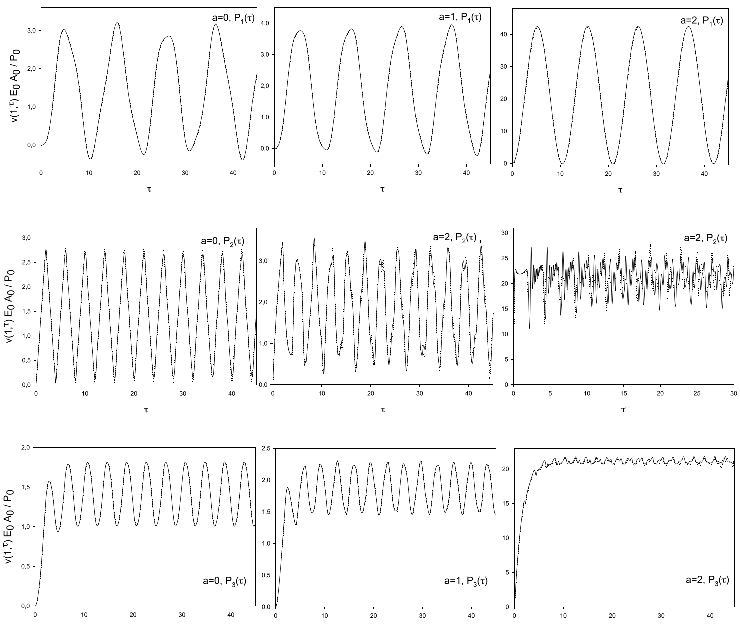


……*CFM* , —— *ANSYS*

Figure 4. Non-dimensional end displacements for case 3 under dynamic loadings.

CONCLUSION

In this work, forced vibration analysis of the inhomogeneous rod with non-uniform cross-section is presented. The material properties and cross-sectional area of the rods are varying continuously in the axial direction. Under the Laplace transformation, the partial differential equation is transformed into time-independent boundary-value problem in spatial domain, which is solved by CFM. The CFM method converts the problem to a system of initial-value problem, which can be solved by any standard methods in the literature. The system of the initial value problem is solved by the fifth-order Runge-Kutta method. Inverse transformation of the results into the time is taken by the modified Durbin's method. From the results presented above, the following conclusions are reached:



……*CFM* , —— *ANSYS*

Figure 5. Non-dimensional end displacements for case 4 under dynamic loadings.

- The solution procedure can be applied to any choice of material and cross-sectional area model. The solution procedure, besides satisfying accuracy with small computational costs, is well structured, simple, and efficient.
- With this combined approach, the forced vibration response is directly obtained. Determination of natural frequencies and mode shapes is not necessary.
- The inhomogeneity parameter is a useful parameter from the design perspective and can be tailored for specific applications so that the displacement amplitude can be controlled.
- The interchange of the expressions defining the material and geometrical properties given in cases yields the same result.
- Furthermore, studies demonstrated that, compared with other numerical methods like FEM, CFM can find highly precise numerical results in less time with a little cost of calculation.

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Appendix A. Modified Durbin's inverse Laplace transform method

A numerical inverse Laplace transform technique is necessary to obtain the values in the time domain. For this purpose, Durbin's inverse Laplace transform technique based on the fast Fourier transform (FFT) is used by Durbin. Durbin's formulation for inverse Laplace transform is summarized as follows:

$$f(t_j) \cong \frac{2e^{aj\Delta t}}{T}\left[-\frac{1}{2}Re\{\bar{F}(a)\} + Re\left\{\sum_{k=0}^{N-1}(\bar{F}(s_k)L_k)\, e^{\left(i\frac{2\pi}{N}\right)jk}\right\}\right],\ (j=0,1,2,..........,N-1) \tag{A1}$$

in which $s_k = a + ik\frac{2\pi}{T}$, $N = \frac{T}{\Delta T}$ where s_k is the kth Laplace transform parameter. T is the solution interval and Δt is the time increment. The selection of constant "a" in numerical inverse Laplace transforms is explained in Durbin (1974)It is implied that if the value of "aT" is chosen in the range 5–10, good results are obtained. Therefore, for the numerical examples presented in this paper, the value of "aT" is generally taken as "6". Finally, results can be modified by multiplying each term of Lanczos factors to obtain better results in the Laplace domain as suggested by Narayanan (1979).

$$L_k = \frac{sin(\frac{k\,\pi}{N})}{(\frac{k\,\pi}{N})} \tag{A2}$$

when $k = 0,\ L_0 = 1.$

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تحليل الاهتزاز القسري للقضبان
غير المتجانسة مع مقطع عرضي غير منتظم

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الخلاصة

تم حل مسألة تحليل الاهتزاز القسري لقضبان الكابولي ذو الخصائص المادية ومناطق المقطع العرضي والتي تختلف عشوائياً في الاتجاه المحوري باستخدام تحويل لابلاس (Laplace) في المجال الزمني وطريقة الدوال التكميلية (CFM) في المجال المكاني. فوفقاً لتحويل لابلاس، تم تحويل المعادلة التفاضلية الجزئية إلى مسألة القيمة الحدية الزمن - المستقل في اتجاه محوري وتم حلها بواسطة CFM. بعد ذلك، تم إجراء التحويل المعكوس باستخدام طريقة Durbin المُعدلة في النطاق الزمني. وفي النهاية، تمت مقارنة نتائج الإزاحة عديمة الأبعاد مع نتائج طريقة FEM المنشورة في الأبحاث. إن النهج المقدم في هذه الدراسة منظم بشكل جيد وبسيط وفعال بالإضافة إلى أنه يوفر قدر كبير من الدقة وتكاليف حسابية قليلة.