

Unification of mathematical concepts and algorithms of k-out-of-n system reliability: A perspective of improved disjoint products

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ABSTRACT

The k-out-of-n system model is the most prominent model of coherent system reliability, with a variety of important special cases, generalizations, and extensions thereof. In particular, the k-out-of-n reliabilities ($1 \leq k \leq n$) constitute a basis for expressing the reliability of an n-order coherent system in terms of its signature (destruction spectrum). A notable algorithm for computing the reliability of a k-out-of-n system is the Improved Disjoint Products (IMDP) algorithm. This paper has four goals, namely, (a) to present a detailed and novel exposition of the IMDP algorithm; (b) to demonstrate that the IMDP algorithm is derivable from the BH-2 algorithm, which is an enhancement of the BH-1 algorithm that is used for evaluating the probability of exactly k successes among n Bernoulli trials and, hence, for computing the probability mass function (pmf) of the generalized binomial distribution; (c) to demonstrate that the IMDP algorithm can be derived from the AR algorithm, which is the Reduced-Ordered-Binary-Decision-Diagram (ROBDD) algorithm for evaluating the k-out-of-n reliability and also for computing the Cumulative Distribution Function (CDF) of the generalized binomial distribution; and (d) to show that the IMDP algorithm is a collective orthogonalization (disjointness) algorithm for a shellable sum-of-products formula (DNF) for k-out-of-n success. The paper plays a unifying role for a variety of concepts and algorithms and tries to emphasize similarities and interrelations among them, while pinpointing any subtle differences among them. A common denominator in explaining the various algorithms is the use of signal flow graphs that are compact, regular, and acyclic. For these loopless graphs, the gain formula requires only simple path enumeration, as well as a calculation of the transmittances of the paths.

Keywords: k-out-of-n; reliability; improved disjoint products; the AR algorithm; the BH-2 algorithm; shellability; signature; unification; signal flow graphs.

NOMENCLATURE

Probability-Ready Expression (PRE): An expression that is directly convertible, on a one-to-one basis, to the corresponding *probability transform* (Rushdi & Ghaleb, 2014). In a PRE,

all ORed terms (products) are disjoint;

all AND ed terms (sums) are statistically independent.

The conversion from a PRE to a probability expression is trivially achieved by replacing Boolean variables by their expectations, AND operations by multiplications, and OR operations by additions (Rushdi, 1983; Rushdi and Goda, 1985; Rushdi and AbdulGhani, 1993; Rushdi and Ba-Rukab, 2005a; Rushdi & Rushdi, 2016). Most of the discussions in this paper pertain to various methods for converting a general switching expression into a PRE, with a stress on achieving property (a) above while preserving property (b).

Duality: The dual of a switching function is obtained by complementing the function and all its switching arguments (inverting both inputs and outputs) (Muroga, 1979; Rushdi, 1993; 2010).

Monotone: A monotone system is one whose reliability function is a non-decreasing function in each component reliability, that is,

$$R(\mathbf{p} | 1_m) - R(\mathbf{p} | 0_m) = \partial R(\mathbf{p}) / \partial p_m \geq 0.0, \quad 1 \leq m \leq n. \quad (1)$$

Here, $R(\mathbf{p} | j_m)$ is $R(\mathbf{p})$ when p_m is set to j where $j = 0, 1$.

Relevant: Component number m is relevant to the system if there exists a valid value for p such that $\partial R(\mathbf{p}) / \partial p_m \neq 0.0$. Relevancy means that $R(\mathbf{p})$ is not vacuous in (independent of) p_m .

Coherent: A coherent system is a monotone system whose components are all relevant (Bergman, 1985). If the reliability function $R(p)$ of a coherent system with equal-reliability components is plotted versus p within the square $0.0 \leq p \leq 1.0, 0.0 \leq R(p) \leq 1.0$, then it satisfies $R(0.0) = 0.0$, and $R(1.0) = 1.0$, and exhibits an S-shape; that is, the curve $R(p)$ versus p is monotonically non-decreasing and if it crosses the diagonal (p versus p), it does so only once and from below (Barlow & Proschan, 1996; Kaufmann et al., 1977, Rushdi & Hassan, 2015; 2016a).

Shellability: A shelling of the sum-of-products formula (DNF)

$$\bigvee_{k=1}^m C_k \quad (2)$$

is a permutation $(C_{\pi(1)}, C_{\pi(2)}, \dots, C_{\pi(m)})$ of its terms such that, for each $k = 1, 2, \dots, m$, the expression

$$\bar{C}_{\pi(1)} \bar{C}_{\pi(2)} \dots \bar{C}_{\pi(k-1)} C_{\pi(k)} \quad (3)$$

is equivalent to an elementary conjunction or to 0. A DNF is called shellable if it admits a shelling (Crama & Hammer, 2011).

Shadow: Let A_1, A_2, \dots, A_m be an ordered list of subsets of $\{1, 2, \dots, n\}$. For $k = 1, 2, \dots, m$, the shadow of A_k depends on the order in which these subsets are listed and is given by the set $S(A_1, A_2, \dots, A_k) = \{j \in \{1, 2, \dots, n\} : \text{there exists } \ell < k \leq m \text{ such that the set difference } A_\ell / A_k = \{j\}\}$ (Crama & Hammer, 2011).

INTRODUCTION

The k-out-of-n:G (F) system, introduced by Birnbaum et al. (1961) more than half a century ago, is a system of n components that functions (fails) if at least k out of its components function (fail) (Misra, 1992; Rushdi, 1993; 2010; Rushdi & Alturki, 2015). Situations in which this system serves as a useful model are frequently encountered in practice. The k-out-of-n:G system covers many interest-

ing systems as special cases. It is also a subclass of many important systems such as threshold systems (Rushdi, 1990; Rushdi & Alturki, 2015) and voting systems (Alturki & Rushdi, 2016). While virtually all nontrivial network reliability problems are known to be NP-hard for general networks (Agrawal & Barlow, 1984), the regular structure of the k-out-of-n system allows the existence of efficient simple algorithms for its reliability analysis that are of quadratic-time linear-space complexity in the worst case (Barlow & Heidtmann, 1984; Zhegalov, 1986; Rushdi, 1986; 1990; 1991; 1993; 2010; Kuo & Zuo, 2003; Amari et al., 2008). Better temporal complexity is also possible via a utilization of the Fast Fourier Transform (FFT) (Belfore, 1995).

The k-out-of-n system plays a central role for the general class of coherent systems, as it can be used to express or approximate the reliability of such systems. In fact, the reliability $R_c(n, \mathbf{p})$ of any coherent system of n components is a weighted sum of the reliabilities $R(k, n, \mathbf{p})$ of k-out-of-n:G system reliabilities (of the same number and reliabilities of components), namely, (Marichal & Mathonet, 2013; Marichal *et al.*, 2011).

$$R_c(n, \mathbf{p}) = \sum_{i=1}^n s_i R(n-i+1, n, \mathbf{p}) \tag{4}$$

Here, $\mathbf{p} = \mathbf{p}_n$ is the vector of component reliabilities $[p_1 \ p_2 \ \dots \ p_n]^T$. The weights s_i are called the system signatures or the destruction spectrum (Samaniego, 1985; 2007; Kochar *et al.*, 1999; Boland, 2001; Boland & Samaniego, 2004a; 2004b; Boland et al., 2003), where.

$$s_i \geq 0 \text{ for all } i, \tag{5}$$

and

$$\sum_{i=1}^n s_i = 1 \tag{6}$$

The signature s_i is the probability $P(T = T_{i:n})$, where T is the time to failure of the system and $T_{i:n}$ is the *i*th order statistic (i.e., the *i*th smallest value) among the component times to failures T_1, T_2, \dots, T_n , assumed independent and identically distributed. Alternatively, s_i might be viewed as the ratio of the number of component orderings for which the *i*th component failure causes system failure to the total number of possible such orderings, which is n!. Note that the signature vector for the (n-k+1)-out-of-n:G {the k-out-of-n:F} system is an n-element vector whose elements are all 0s except the *k*th element, which is 1 (in immediate agreement with the definition of such a system). This means that equation (4) becomes a self-consistent identity if the coherent system considered is a k-out-of-n system.

Samaniego (2007) explains how system signatures are computed (somewhat inefficiently) *via* the basic definition. He also presents a better algorithm for computing signatures *via* the domination theory of Satyanarayana and Prabhakar (1978) using the concept of formations that considerably simplifies the Inclusion-Exclusion Principle. An explicit formula for signatures is given by Boland (2001).

$$s_i = r_{n-i+1}(n) / \binom{n}{n-i+1} - r_{n-i}(n) / \binom{n}{n-i}, \tag{7}$$

where $r_i(n)$ is the number of primitive system paths of cardinality i, that is, the number of

primitive states of the system (or minterms of system success) in which exactly i components are functioning (and hence exactly $(n - i)$ components are failed). Here, $\binom{n}{k}$ denotes the combinatorial (binomial) coefficient, that is, the number of ways of selecting k out of n objects without order or repetition.

Typically, the signature vector of a coherent system starts and terminates with some zero elements; that is, it takes the form.

$$s = [0 \dots 0 s_f \dots s_l, 0 \dots 0]^T, \quad (8)$$

where $1 \leq f \leq l \leq n$, and s_f and s_l denote the locations of the *first* and *last* nonzero elements of the signature vector s . Since the k -out-of- n :G reliabilities are ordered via.

$$R(i, n, \mathbf{p}) \geq R(j, n, \mathbf{p}) \text{ for } 1 \leq i \leq j \leq n, \quad (9)$$

relations (4), (6), and (9) can be combined to yield the following lower and upper bounds for $R_c(n, \mathbf{p})$.

$$R(n-f+1, n, \mathbf{p}) \leq R_c(n, \mathbf{p}) \leq R(n-l+1, n, \mathbf{p}). \quad (10)$$

Note that (10) obtains bounds on $R_c(n, \mathbf{p})$ rather than an exact value for it as (4) does. However, it does not demand the computation of signatures. It only requires a determination of f and l which can be readily obtained *via* some partial information about the minimal cutsets, which are the prime implicants of system failure, and about the minimal pathsets, which are the prime implicants of system success. In fact f can be identified as the cardinality of a system minimal cutset that has the smallest number of components, while $(n-l+1)$ is identified as the cardinality of a system minimal pathset that has the smallest number of components.

One purpose in this paper is to present a detailed exposition of a notable algorithm for computing the reliability of a k -out-of- n :G system, namely, the Improved Disjoint Products (IMDP) algorithm, first outlined briefly by Rushdi (1993). This algorithm extends the basic idea of disjoint products (Abraham, 1979; Dotson & Gobien, 1979; Rushdi, 1993; Rushdi & Al-khateeb, 1983; Ball & Provan, 1988; Barlow & Iyer, 1988). This exposition is extended to show that the IMDP algorithm is a special case of each of the following: (a) the AR algorithm (Rushdi, 1986; 1990; 1991, 1993, 2010; Kuo & Zuo, 2003); (b) the BH-2 algorithm (Barlow & Heidtmann, 1984; Rushdi, 1993); and (c) an orthogonalization (disjointness) algorithm of a shellable formula for k -out-of- n success (Crama & Hammer, 2011). The paper's main goal is to unify and interrelate many seemingly disparate concepts and set the stage for a comprehensive theory for k -out-of- n system reliability. The paper relies heavily on the use of Mason Signal Flow Graphs (SFGs) (Golnaraghi & Kuo, 2009) for offering pictorial explanations, a practice that has been used frequently in the reliability literature (Rushdi, 1986; 1987; 1990; 1993; 2010; Rushdi & Dehlawi, 1988; Rushdi & Althubaiti, 1993; Kuo & Zuo, 2003; Rushdi & Alturki, 2015). The SFG pictorial explanation is an equivalent alternative to the visual interpretation typically given to the Reduced Ordered Binary Decision Diagram (ROBDD), first introduced by Bryant (1986) and utilized in reliability studies by many researchers including Rauzy (2008), Pock *et al.* (2011); Mo (2014); Mo *et al.* (2014; 2015); and Li *et al.* (2014).

Several observations about the scope of this paper are in order.

- The paper deals with a classical problem of system reliability in which system reliability $R(k, n, \mathbf{p})$ is related to component reliabilities \mathbf{p} . If a component is known in terms of its *failure rate* λ_i , then the component reliability is $p_i(t) = \exp(-\lambda_i t)$. Hence, our study herein is trivially extended to express $R(k, n, \mathbf{p})$ as a function of component failure rates.
- The present study deals with systems of *heterogeneous* components. If, instead, the system has homogeneous components of independent and identically distributed failure rates, then all the component reliabilities p_i become equal to a common value p . In this case, the present algorithms are not warranted since system reliability can be expressed in terms of the Cumulative Distribution Function (CDF), or the probability mass function (pmf) of the binomial probability distribution (Rushdi, 1993; Rushdi & Al-Qasimi, 1994; Al-Qasimi & Rushdi, 2008; Rushdi, 2010).
- This paper uses the basic model for studying k -out-of- n systems, in which component reliabilities are assumed constant and the time parameter is implicit rather than explicit in the analysis. A more realistic model called the load-sharing model takes into consideration the fact that when $(n - k)$ or less components fail, the system remains working with an increase in the load on the remaining components, thereby increasing the failure rates (and hence decreasing the reliabilities) of these components. Such a load-sharing model makes an explicit use of time via Markov analysis (Scheuer, 1988; Shao & Lamberson, 1991; Liu, 1998; Kvam & Pena, 2005; Amari et al., 2008; Amari & Bergman, 2008; Levitin et al., 2012).
- The failure rates of system components are typically estimated via statistical techniques and, hence, are known only with *uncertainty*. Such uncertainty in component reliabilities propagates into uncertainty in system reliability. This issue is not addressed herein, and the reader is referred to earlier work (Rushdi, 1985; Rushdi & Ba-Rukab, 2005a; 2005b; Bamasak & Rushdi, 2016; Rushdi & Hassan, 2016b) for an assessment of the uncertainty in system reliability in terms of the uncertainties in component reliabilities.
- The paper addresses the *binary* model rather than the multi-state model of a k -out-of- n system. It unifies concepts of many prominent k -out-of- n algorithms for the binary case and sets the stage for a similar treatment of the multi-state case.
- The paper has a unique way of *pictorially* presenting the pertinent algorithms via Signal Flow Graphs, which are all acyclic (loopless) and beautifully regular and symmetric. The symmetry of these graphs is stressed by using the symmetric coordinates $k_1 = k$ and $k_2 = n - k$. The choice of these coordinates restricts graph edges to be only horizontal or vertical, with no oblique edges such as these in earlier work (Rushdi, 1986; 1993, 2010) where the usual coordinates k and n are used.
- The paper offers a single explicit running example or case study, namely, that of the 4-out-of-7:G system. However, the pictorial nature of the SFG representations enables the reader to visualize additional case studies for any smaller system. For example, the 2-out-of-3:G system can be visualized by viewing the node for $k_1 = 2$ and $k_2 = 1$ in all forthcoming graphs.
- The paper original contribution is a novel mathematical derivation of the IMPDP algorithm as well as novel proofs that this algorithm is derivable from each of the BH2 and AR algorithms. The paper also contributes novel unification of many apparently disparate concepts of reliability theory.

The practical importance of the k-out-of-n model cannot be overestimated (Kuo and Zuo, 2003). Rushdi (2010) lists many engineering applications of this model as such, and of the variety of algorithms discussed herein. Notable among these applications are secure data communication in mobile ad hoc networks (Papadimitratos & Haas, 2006), small-fleet aircraft redundancy and spares (Cochran & Lewis, 2002), oil-supply systems (Tian *et al.*, 2008), n-version programming (Yamachi & Yamamoto, 2006; Hiroshima *et al.*, 2006), furnace systems (Zuo *et al.*, 1999), power systems (Rushdi, 1990), ecological corridors (Rushdi & Hassan, 2015; 2016a), converter valves for power transmission (Wang *et al.*, 2010), and multiprocessor systems (Milad *et al.*, 2012).

The organization of the rest of this paper is as follows. This introduction is preceded by a summary of some of the nomenclatures used and followed by a formal derivation and exposition of the IMPD method, which enhances and updates its brief outline in Rushdi (1993). Next, we briefly outline the BH-2 algorithm, which is a quadratic-time algorithm that is occasionally inadvertently confused with the AR algorithm. The BH-2 algorithm is related to the BH-1 algorithm, which computes the probability of k successes among n generalized Bernoulli trials, or equivalently, the probability mass function (pmf) of the generalized Bernoulli distribution. We also derive the IMPD algorithm from the SFG underlying the BH-2 algorithm. Next, we present a quick review of the AR algorithm, which is a quadratic-time algorithm for evaluating the k-out-of-n reliability, or, equivalently, for computing the Cumulative Distribution Function (CDF) (rather than the pmf) of the generalized binomial distribution. We also point out that the AR algorithm is the ROBDD algorithm for computing the k-out-of-n reliability and proceed to prove that the IMPD method is derivable from the SFG structure underlying the AR algorithm. Later, we consider the concept of a shellability algorithm, that is, a disjointing algorithm that does not increase the number of products in a sum-of-products expression. Subsequently, We demonstrate that the IMPD algorithm is a collective shellability algorithm in the sense that it orthogonalizes (disjoints) a shellable formula for k-out-of-n success, and it does so in a collective fashion. The paper concludes with some observations and comments.

A FORMAL DERIVATION OF THE IMPD FORMULA

Two dual IMPD formulas for the reliability $R(k, n, \mathbf{p})$ and unreliability $U(k, n, \mathbf{p})$ of a k-out-of-n:G system with component reliabilities \mathbf{p} are reported by Rushdi (1993) in pp. 214-215, following a qualitative description in pp. 197-198 that enhances and organizes earlier work by Heidtmann (1982), and Locks and Heidtmann (1984). In the following, we present a *formal* derivation (starting in the Boolean domain and terminating in the probability domain) of the IMPD formula for $R(k, n, \mathbf{p})$. The success $S(k, j, \mathbf{X}_j)$ of a k-out-of-j:G system is given by its Boole-Shannon expansion about X_j (namely, Equation (5.40a) of Rushdi (1993)):

$$S(k, j, \mathbf{X}_j) = \bar{X}_j S(k, j-1, \mathbf{X}_{j-1}) \vee X_j S(k-1, j-1, \mathbf{X}_{j-1}), \quad 1 \leq k \leq j \leq n, \quad (11)$$

where $\mathbf{X}_j = [X_1 \ X_2 \ \dots \ X_j]^T$ is the vector of the first j component successes. The R.H.S. of equation (11) involves two subfunctions or restrictions $S(k, j-1, \mathbf{X}_{j-1})$ and $S(k-1, j-1, \mathbf{X}_{j-1})$ of the original success function. We express the latter subfunction in terms of the former as follows:

$$\begin{aligned}
 S(k-1, j-1, \mathbf{X}_{j-1}) &= \bigvee_{l=k-1}^{j-1} S_e(l, j-1, \mathbf{X}_{j-1}) \\
 &= S_e(k-1, j-1, \mathbf{X}_{j-1}) \bigvee \bigvee_{l=k}^{j-1} S_e(l, j-1, \mathbf{X}_{j-1}) \\
 &= S_e(k-1, j-1, \mathbf{X}_{j-1}) \bigvee S(k, j-1, \mathbf{X}_{j-1}), \quad 1 \leq k \leq j \leq n.
 \end{aligned} \tag{12}$$

Here, $S_e(l, j, \mathbf{X}_j)$ is the indicator of *exactly* l successes among j trials (exactly of the first j components are good). The set of $(j+1)$ values $S_e(l, j, \mathbf{X}_j) \ 0 \leq l \leq j$, have expectations $E(l, j, \mathbf{p}_j)$ which constitute the pmf of the generalized binomial distribution. Now, we substitute (12) into (11) to obtain

$$\begin{aligned}
 S(k, j, \mathbf{X}_j) &= \bar{X}_j S(k, j-1, \mathbf{X}_{j-1}) \bigvee X_j S_e(k-1, j-1, \mathbf{X}_{j-1}) \bigvee X_j S(k, j-1, \mathbf{X}_{j-1}) \\
 &= (\bar{X}_j \bigvee X_j) S(k, j-1, \mathbf{X}_{j-1}) \bigvee X_j S_e(k-1, j-1, \mathbf{X}_{j-1}) \\
 &= S(k, j-1, \mathbf{X}_{j-1}) \bigvee X_j S_e(k-1, j-1, \mathbf{X}_{j-1}), \quad 1 \leq k \leq j \leq n
 \end{aligned} \tag{13}$$

Equation (13) consists of two disjoint (orthogonal) terms, whose ANDing is 0. Obviously,

$$\begin{aligned}
 &S_e(k-1, j-1, \mathbf{X}_{j-1}) S(k, j-1, \mathbf{X}_{j-1}) \\
 &= S_e(k-1, j-1, \mathbf{X}_{j-1}) \{ \bigvee_{l=k}^{j-1} S_e(l, j-1, \mathbf{X}_{j-1}) \} \\
 &= \bigvee_{l=k}^{j-1} S_e(k-1, j-1, \mathbf{X}_{j-1}) S_e(l, j-1, \mathbf{X}_{j-1}) = \bigvee_{l=k}^{j-1} (0) \\
 &= 0.
 \end{aligned} \tag{14}$$

We now write instances of (13) for j decreasing from n to k {restricted by $1 \leq k \leq j \leq n$ }

$$\begin{aligned}
 S(k, n, \mathbf{X}_n) &= S(k, n-1, \mathbf{X}_{n-1}) \bigvee X_n S_e(k-1, n-1, \mathbf{X}_{n-1}), \\
 S(k, n-1, \mathbf{X}_{n-1}) &= S(k, n-2, \mathbf{X}_{n-2}) \bigvee X_{n-1} S_e(k-1, n-2, \mathbf{X}_{n-2}), \\
 &\dots\dots\dots \\
 S(k, k, \mathbf{X}_k) &= S(k, k-1, \mathbf{X}_{k-1}) \bigvee X_k S_e(k-1, k-1, \mathbf{X}_{k-1}) \\
 &= 0 \bigvee X_k S_e(k-1, k-1, \mathbf{X}_{k-1}) = X_k S_e(k-1, k-1, \mathbf{X}_{k-1}).
 \end{aligned} \tag{15}$$

In equation (15), we made use of the boundary condition (Rushdi, 1993)

$$S(k, k-1, \mathbf{X}_{k-1}) = 0, \quad k \geq 1.$$

In view of (14), the second term in the R.H.S. of each success equation in (15) is disjoint with the first term in the same equation and, hence, is disjoint with *all* terms in successor equations. Substituting each success equation in (15) into its predecessor one, we obtain

$$\begin{aligned}
 S(k, n, \mathbf{X}_n) &= X_n S_e(k-1, n-1, \mathbf{X}_{n-1}) \vee X_{n-1} S_e(k-1, n-2, \mathbf{X}_{n-2}) \vee \dots \vee X_k S_e(k-1, k-1, \mathbf{X}_{k-1}) \\
 &= \bigvee_{j=k}^n X_j S_e(k-1, j-1, \mathbf{X}_{j-1}), 1 \leq k \leq n.
 \end{aligned} \tag{16}$$

Equation (16) is the IMDP formula in the Boolean domain. It is a PRE formula since it consists of disjoint terms that are presumed to be PREs themselves and, hence, translates, on a one-to-one basis, into the probability domain as

$$R(k, n, \mathbf{p}) = R(k, n, \mathbf{p}_n) = \sum_{j=k}^n p_j E(k-1, j-1, \mathbf{p}_{j-1}), 1 \leq k \leq n. \tag{17}$$

Equation (17) is the IMDP formula in the probability domain. It says that $R(k, n, \mathbf{p})$ is a weighted sum of the last $(n-k+1)$ component reliabilities. The weight for component reliability $p_l (k \leq l \leq n)$ is the probability that exactly $(k-1)$ of the preceding components are successful. Note that any particular component reliability $p_l (k \leq l \leq n)$ appears in its own term and then reappears within the weights of successor component reliabilities. The formula can be rewritten as the weighted sum of any ordered set of $(n-k+1)$ component reliabilities, each with an appropriate weight that is equal to the probability that exactly $(k-1)$ of the preceding elements within this set are successful. The numbers of terms in the two sides of (17) correspond to the well-known binomial identity

$$\binom{n}{k} = \sum_{l=k}^n \binom{l-1}{k-1}. \tag{18}$$

THE IMDP ALGORITHM AS A SPECIAL CASE OF THE BH-2 ALGORITHM

Barlow and Heidtmann (1984) presented two algorithms (frequently named the BH-1 and BH-2 algorithms) for computing $R(k, n, \mathbf{p})$. These two algorithms are based solely on the use of generating functions, but have been given a recursive-relation interpretation by Rushdi (1986; 1993; 2010).

Basically, the BH-1 algorithm computes $E(k, n, \mathbf{p})$, $0 \leq k \leq n$, which is the probability mass function (pmf) of the generalized binomial distribution (Rushdi & Al-Qasimi, 1994; Al-Qasimi & Rushdi, 2008) and, hence, is the probability of *exactly* k successes in n trials; that is, it is the expected value of $S_e(k, n, \mathbf{X})$. The BH-1 algorithm computes $R(k, n, \mathbf{p})$ as a summation of $E(j, n, \mathbf{p})$ values, where $k \leq j \leq n$, i. e.,

$$R(k, n, \mathbf{p}) = \sum_{j=k}^n E(j, n, \mathbf{p}), \tag{19a}$$

where $\mathbf{p} = \mathbf{p}_n = [p_1 \ p_2 \ p_3 \ \dots \ p_n]^T$ is the vector of component reliabilities. Equation (19a) amounts to converting the pmf of the generalized Binomial distribution into the complement of the corresponding Cumulative Distribution Function (CDF). The probabilities $E(k, n, \mathbf{p})$ can be obtained via differencing that converts the aforementioned CDF to the corresponding pmf, namely,

$$E(k, n, \mathbf{p}) = R(k, n, \mathbf{p}) - R(k+1, n, \mathbf{p}). \tag{19b}$$

These probabilities are governed by the recursive relation (Rushdi, 1986; 1993)

$$E(k, j, \mathbf{p}_j) = q_j E(k, j-1, \mathbf{p}_{j-1}) + p_j E(k-1, j-1, \mathbf{p}_{j-1}), \quad 0 \leq k \leq j \leq n, \quad j \geq 1, \tag{20}$$

together with the boundary conditions:

$$E(0, 0, \mathbf{p}_0) = 1.0, \quad (21a)$$

$$E(-1, j, \mathbf{p}_j) = 0.0, j \geq 0, \quad (21b)$$

$$E(j+1, j, \mathbf{p}_j) = 0.0, j \geq 0. \quad (21c)$$

Rushdi (1993) presents relations (20)-(21) pictorially in terms of Mason Signal Flow Graphs (SFGs) in the (k, n) plane. We reproduce these SFGs herein in the (k_1, k_2) plane, with $k_1 = k$ and $k_2 = n-k$, so as to *stress their symmetry* with respect to the k_1 and k_2 arguments, and set the stage for extensibility to multi-state systems with k_1, k_2, \dots, k_m ($m > 2$) arguments. Figure 1 illustrates an octant of the (k_1, k_2) plane wherein the recursive relation (Eq. (20)) for $E(k_1, k_1+k_2, \mathbf{p}_{k_1+k_2})$, ($k_1 \geq 0, k_2 \geq 0, (k_1+k_2) \geq 1$) is valid, bounded by condition (21a) at $\{k_1=k_2=0\}$, condition (21b) at $\{k_1=-1, k_2 \geq 1\}$, and condition (21c) at $\{k_1 \geq 1, k_2=-1\}$. Figure 1 is totally and beautifully symmetric around the symmetry axis $k_1 = k_2$. While nodes in Fig. 1 represent $E(k_1, k_2, \mathbf{p}_{k_1+k_2})$, the figure depicts a particular double-circle node representing $R(4, 7, \mathbf{p}_7)$ as the summation of $E(4, 7, \mathbf{p}_7)$, $E(5, 7, \mathbf{p}_7)$, $E(6, 7, \mathbf{p}_7)$ and $E(7, 7, \mathbf{p}_7)$. Each shaded circle node in Fig.1 is given recursively *via* (20) as a weighted sum of the nodes immediately above it and the node immediately to the left of it. Each node on the secondary diagonal $k_1+k_2 = n$ has, therefore, exactly two arrows incident on it, a vertical one of transmittance p_n and a horizontal one of transmittance q_n . Therefore, while nodes on the secondary diagonal $k_1+k_2 = (n-1)$ represent exactly k_1 successes out of $(n-1)$ trials, they can simply be augmented by the n^{th} trial (of success p_n and failure q_n) to produce nodes on the next secondary diagonal $k_1+k_2 = n$. Square nodes are not expressed recursively, since they represent boundary conditions and possess specific values; white nodes are of value 0.0, while the single black node is of value 1.0.

Now, we note that equation (13) is in PRE form (thanks to the orthogonality (disjointness) between $S(k, j-1, \mathbf{X}_{j-1})$ and $S_e(k-1, j-1, \mathbf{X}_{j-1})$ in (14)), and hence it has the probability domain counterpart

$$R(k, j, \mathbf{p}_j) = R(k, j-1, \mathbf{p}_{j-1}) + p_j E(k-1, j-1, \mathbf{p}_{j-1}), \quad 1 \leq k \leq j \leq n. \quad (22)$$

Rushdi (1986; 1993) proved that (22) was implicit in the transformation of the BH-1 algorithm to the BH-2 algorithm. This fact is demonstrated by Fig. 2, in which the 4-out-of- $(4+k_2)$:G reliabilities ($0 \leq k_2 \leq 3$) are computed *via* (22) and are represented by double-circle nodes. Similarly to Fig. 1, Fig. 2 uses single-circle nodes to represent the pmf probabilities $E(k_1, k_1+k_2, \mathbf{p}_{k_1+k_2})$, while the four double-circle nodes represent the CCDF probabilities (i.e., reliabilities) $R(4, 4+k_2, \mathbf{p}_{4+k_2})$, $0 \leq k_2 \leq 3$.

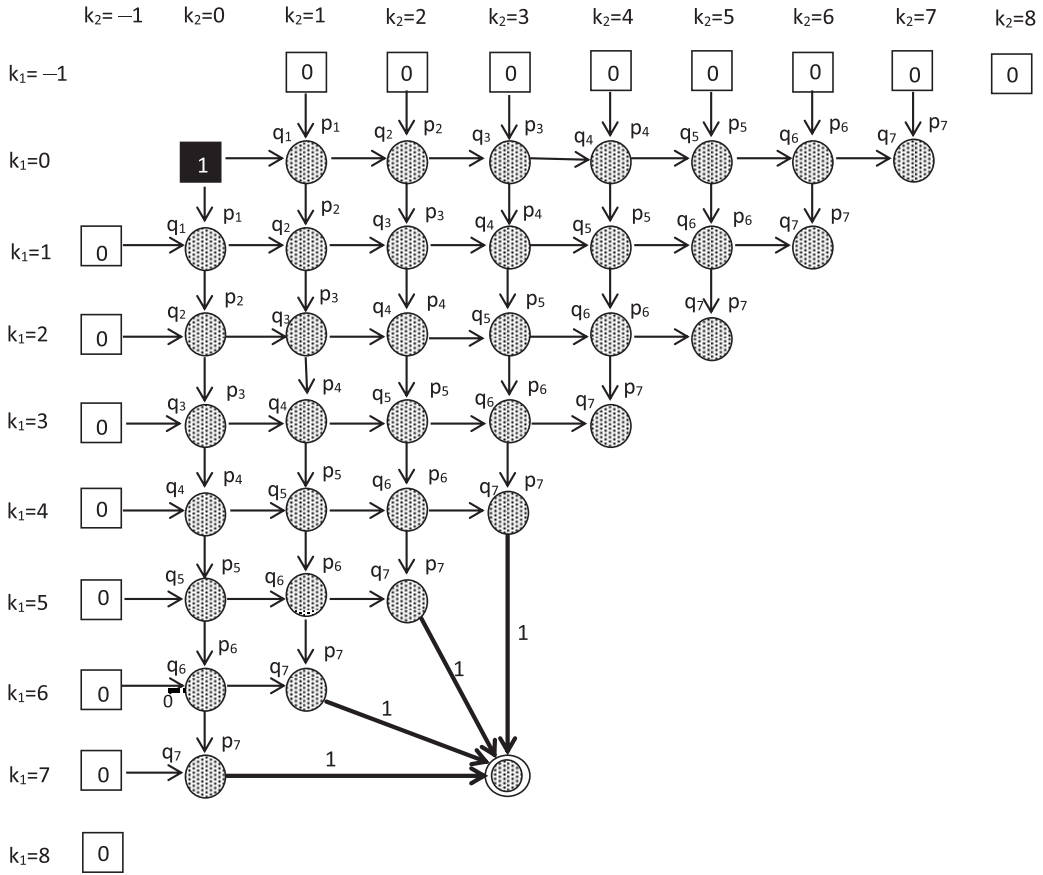


Fig. 1. The Signal Flow Graph for computing the probability of exactly k_1 successes out of (k_1+k_2) trials up to $(k_1+k_2) = 7$, augmented by thick arrows to show the BH-1 algorithm for $R(4, 7, \mathbf{p}_7)$.

We now demonstrate that the IMDP algorithm is derivable from the BH-2 algorithm (in fact, equation (13) was crucial for both algorithms). Figure 2 suggests that $R(4, 7, \mathbf{p}_7)$ can be computed just by summing the transmittances from the single source at $\{k_1 = k_2 = 0\}$ to the $R(4, 7, \mathbf{p}_7)$ node, thanks to the fact that the Signal Flow Graph in Fig. 2 is acyclic (i.e., has no loops). The transmittances terminating with p_4, p_5, p_6 and p_7 are disjoint and hence $R(4, 7, \mathbf{p}_7)$ can be written as a weighted sum of the component reliabilities of p_4, p_5, p_6 and p_7 . The weights or coefficients for these component reliabilities are deduced as the nodes $k_1 = 3$ ($0 \leq k_2 \leq 3$) in Fig. 2 as shown in Fig. 3, and result in the formula

$$\begin{aligned}
 R(4, 7, \mathbf{p}_7) &= p_4 E(3, 3, \mathbf{p}_3) + p_5 E(3, 4, \mathbf{p}_4) + p_6 E(3, 5, \mathbf{p}_5) + p_7 E(3, 6, \mathbf{p}_6) \\
 &= \sum_{l=4}^7 p_l E(3, l-1, \mathbf{p}_{l-1}).
 \end{aligned}
 \tag{23}$$

Note that Fig. 3 is simply a collection of four sub-figures obtained by terminating Fig. 2 at the output nodes $E(3, 3, \mathbf{p}_3)$, $E(3, 4, \mathbf{p}_4)$, $E(3, 5, \mathbf{p}_5)$, and $E(3, 6, \mathbf{p}_6)$, respectively, that is, by omitting paths of Fig. 2 that are *irrelevant* to the computation of each of these values. Equation (23) is the special case $\{k = 4, n = 7\}$ of the general IMDP formula (17). Note that equation (23) arranges the $\binom{7}{4} = 35$

terms of $R(4, 7, \mathbf{P})$ as $\binom{3}{3} = 1$, $\binom{4}{3} = 4$, $\binom{5}{3} = 10$, $\binom{6}{3} = 20$ terms, respectively. Extension of the current SFG derivation for general k and n is straightforward and obvious.

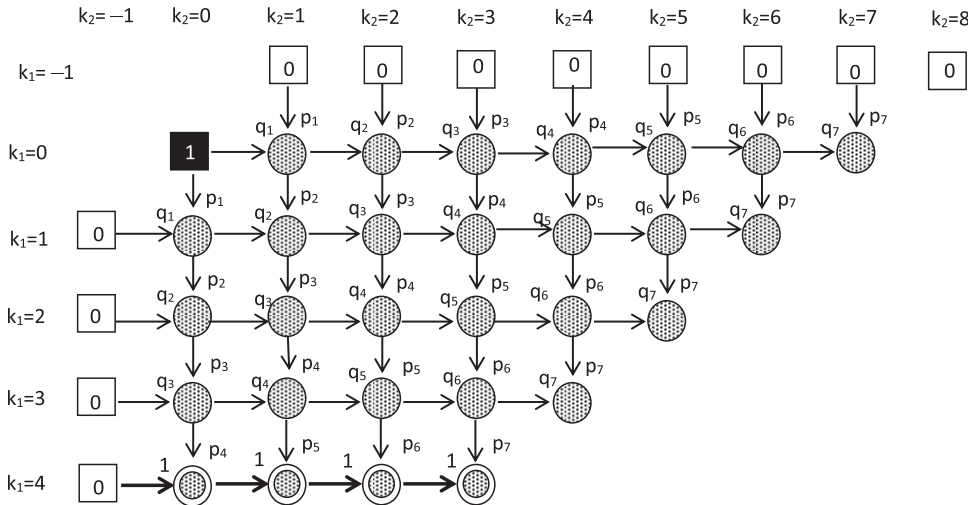


Fig. 2. The Signal Flow Graph for computing the 4-out-of- $(4+k_2)$: G reliabilities via the BH-2 algorithm ($0 \leq k_2 \leq 3$).

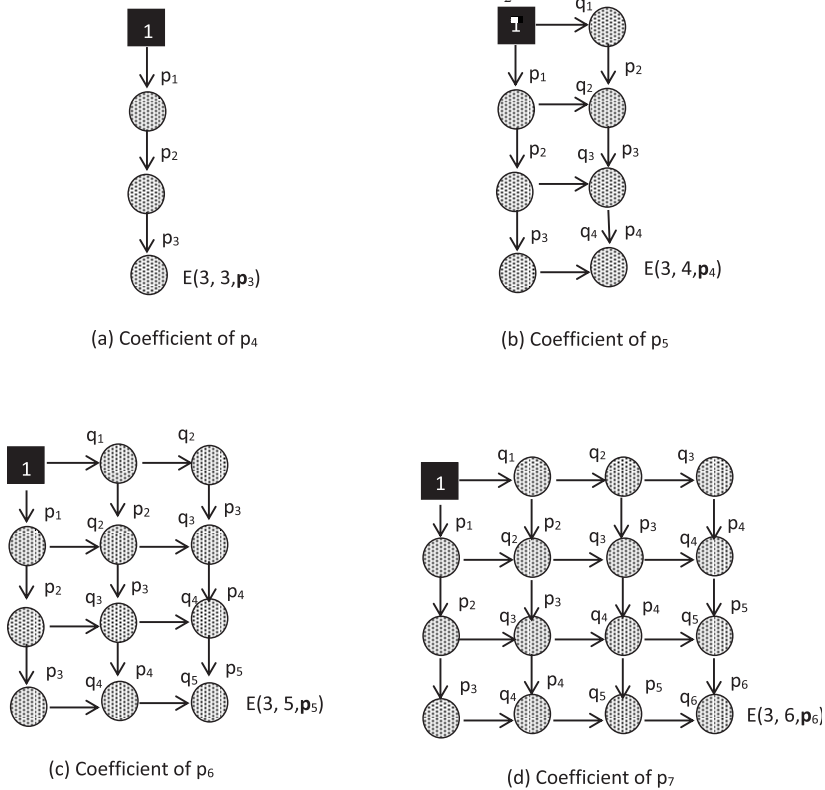


Fig. 3. A representation of the coefficient of p_l ($4 \leq l \leq 7$) in the expression of $R(4, 7, \mathbf{p})$ via the BH-2 algorithm. These coefficients are naturally displayed as E values, that is, single-circle nodes.

THE IMPDP ALGORITHM AS A SPECIAL CASE OF THE AR ALGORITHM

The AR algorithm is a quadratic-time *iterative* algorithm that was developed by Rushdi (1986) and expounded in Rushdi (1990; 1991; 1993; 2010), Rushdi & Al-Thubaity (1993), Rushdi & Al-Hindi (1993), and Kuo & Zuo (2003) and later reappeared, *in disguise*, in Wu & Chen (1994) and Dutuit & Rauzy (2001). The algorithm has the beautiful characteristic of having the same complexity for computing both the reliability and unreliability of either the k-out-of-n:G system or the k-out-of-n:F system. It is also an efficient and direct algorithm for computing the Cumulative Distribution Function (CDF) of the generalized binomial distribution (Rushdi & Al-Qasimi (1994); Al-Qasimi & Rushdi (2008)). It was pointed out in Rushdi (2010) that the AR algorithm is, in fact, an implementation of the Reduced-Ordered-Binary-Decision-Diagram (ROBDD) strategy when this strategy is adapted for computing the k-out-n-reliability. The ROBDD strategy was proposed by Bryant (1986) as an extension of the BDD methodology of Akers (1960). The ROBDD deals with *general* switching (two-valued Boolean) functions and is now considered the state-of-the-art data structure for handling such functions, with extensive applications in reliability (Rauzy, 2008; Misra, 2008; Bjorkman, 2013; Mo, 2014). The AR algorithm, however, handles a class of switching functions that are both *monotonically non-decreasing* and *totally symmetric*. Apart from this restriction in applicability, the AR algorithm has exactly the same features as the ROBDD algorithm, namely, as follows:

1. Both the AR and ROBDD algorithms are based on the Boole-Shannon expansion in the Boolean domain

$$f(\mathbf{X}) = \overline{X_j} (f(\mathbf{X} / \overline{X_j}) \vee X_j (f(\mathbf{X}) / X_j), \tag{24}$$

where

$$f(\mathbf{X}) / \overline{X_j} = f(\mathbf{X}) \rfloor X_j = 0, \tag{25a}$$

$$f(\mathbf{X}) / X_j = f(\mathbf{X}) \rfloor X_j = 1, \tag{25b}$$

are called quotients, ratios, cofactors, subfunctions, or restrictions of $f(\mathbf{X})$. This expansion translates in the probability domain to the following expression for system reliability.

$$R(\mathbf{p}) = q_j R(\mathbf{p} \mid p_j=0) + p_j R(\mathbf{p} \mid p_j=1), \tag{26}$$

where $q_j = 1.0 - p_j$. Our earlier equation (11) is the particular instance of (24) when applied to the k-out-of-n:G success. Equation (26) is simply an expression of the Total Probability Theorem (Trivedi, 2002; Rushdi & Rushdi, 2016) or the Factoring Theorem (Satyanarayana & Chang, 1983; Rushdi & Hassan, 2015; 2016a).

2. Both algorithms visit the variables in a certain order, typically monotonically ascending or monotonically descending.

3. Both algorithms reduce the resulting expansion tree (which is exponential in size) to a rooted acyclic graph that is both canonical and hopefully compact or sub-exponential. The reduction rules (Mo et al., 2015) require 3(a) merging isomorphic subtrees and 3(b) deletion of useless nodes whose outgoing edges point to the same child node.

Figure 4 depicts a Mason Signal Flow Graph (SFG) for computing all k -out-of- n : G reliabilities for $1 \leq k \leq n = 7$. Similarly to earlier figures, Figure 4 is drawn over a rectangular lattice of coordinates $k_1 = k$, $k_2 = n - k$. For convenience in forthcoming computations, we use a reverse order (from 7 down to 1, instead of a forward order from 1 to 7) for the component reliabilities. This graph is the essence of the *iterative* AR algorithm and helps the reader visualize the basic recursive relations used by the AR algorithm, which are a special case of (26), namely,

$$R(k, j, \mathbf{p}_j) = q_j R(k, j-1, \mathbf{p}_{j-1}) + p_j R(k-1, j-1, \mathbf{p}_{j-1}), \quad 1 \leq k \leq j \leq n, \quad (27)$$

These recursive relations should be used together with the boundary conditions

$$R(0, j, \mathbf{p}_j) = 1.0, \quad j \geq 0, \quad (28a)$$

$$R(j+1, j, \mathbf{p}_j) = 0.0, \quad j \geq 0. \quad (28b)$$

Note that a node in Figure 4 with coordinates k_1 and k_2 represents the k_1 -out-of- $(k_1 + k_2)$: G reliability and, hence, is represented by a double circle, in conformance with the notation in earlier figures. Similarity to earlier figures, shaded circles represent nodes expressed recursively (this time *via* (27)), while squares represent boundary conditions of specific values 0.0 and 1.0 for white and black nodes, respectively. There is a striking similarity between Fig. 4 and Fig.1 because they share the same *recursive structure* due to the similarity of equations (22) and (27). However, we stress on the fact that these two figures represent *different entities* and have obviously different *boundary conditions* as well as different *regions of validity* for the common recursive structure. For comparison, Fig. 5 depicts the individual ROBDDs used in the computation of $R(k, 7, \mathbf{p})$, $1 \leq k \leq 7$, with each being isomorphic to the pertinent subgraphs in Fig. 4. In an ROBDD, a node does not represent a reliability value but depicts a *decision point* and hence is designated by the pertinent decision variable. The two outgoing edges of each node are labeled by 0 and 1 representing the two states of the decision variable (analogously to the (\overline{X}_i, X_i) or (q_i, p_i) transmittances on the SFG). While the diagrams in Figures 4 and 5 are definitely equivalent, the one in Fig. 4 more readily allows a human user to construct symbolic or numerical reliability expressions, while the one in Fig. 5 describes an automated code obtaining such symbolic or numeric values. In passing, we stress that we deliberately added unnecessary (albeit helpful) features to the ROBDDs in Fig. 5, such as using double circles for the recursive nodes, colouring the leaf nodes of values 0 and 1 as white and black, respectively, and using split leaf nodes of 0 and 1 (rather than a single leaf node of value 0 and a single leaf node of value 1). With these added features, the analogy between the lumped Fig. 4 and the individual subfigures in Fig. 5 becomes clearer and more apparent.

We now demonstrate that the IMDP algorithm is derivable from the AR algorithm. Figure 4 suggests that the sink node $R(4, 7, \mathbf{p}_7)$ (the node at $k_1 = 4$ and $k_2 = 3$) can be computed as a *superposition* of contributions of the four unity nodes at $k_1 = 0$ and $0 \leq k_2 \leq 3$. Figure 4 indicates clearly that the rest of the unity nodes on the line $k_1 = 0$ contribute nothing to the sink node $R(4, 7, \mathbf{p}_7)$, because no path emanating from any of them can reach the node at $k_1 = 4$ and $k_2 = 3$. Again, the Signal Flow Graph in Figure 4 is acyclic and has no loops. Consequently, the individual contribution of each source node at $k_1 = 0$ and $0 \leq k_2 \leq 3$ to $R(4, 7, \mathbf{p}_7)$ is the transmittance from that source node to the node at $k_1 = 4$ and $k_2 = 3$. Such a contribution is equal to p_{7-k_2} ($0 \leq k_2 \leq 3$) multiplied by a certain coefficient that equals the transmittance to the sink node from the node below the actual source node, that is, the one at $k_1 = 1$ ($0 \leq k_2 \leq 3$), and hence is represented by the subfigures of Figure 6.

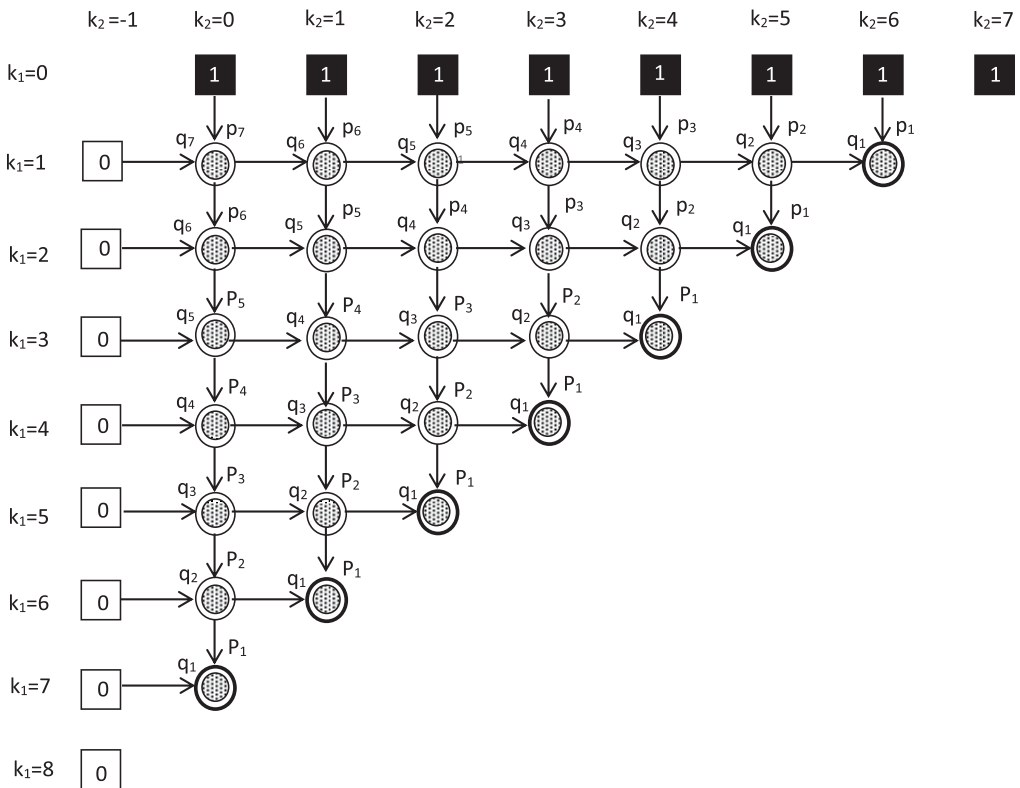
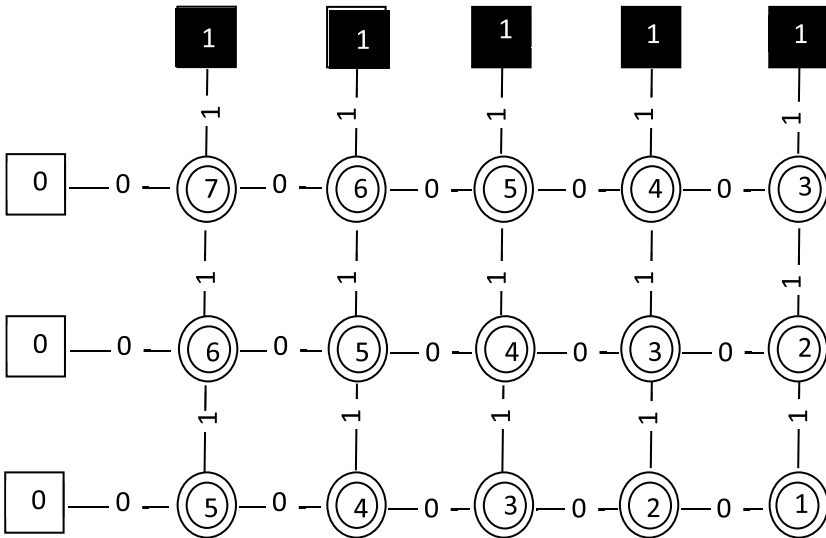
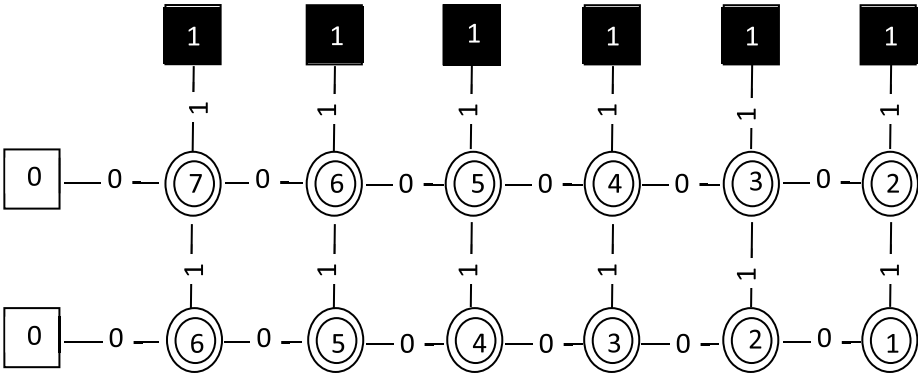
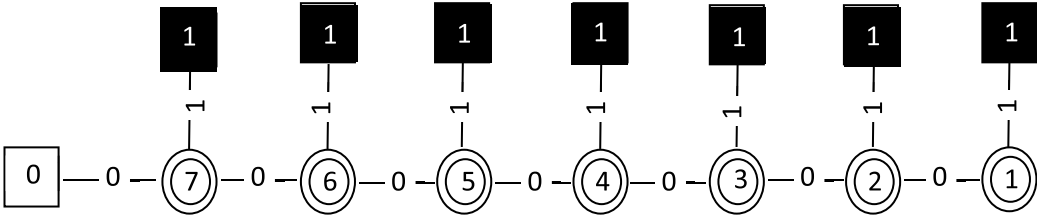
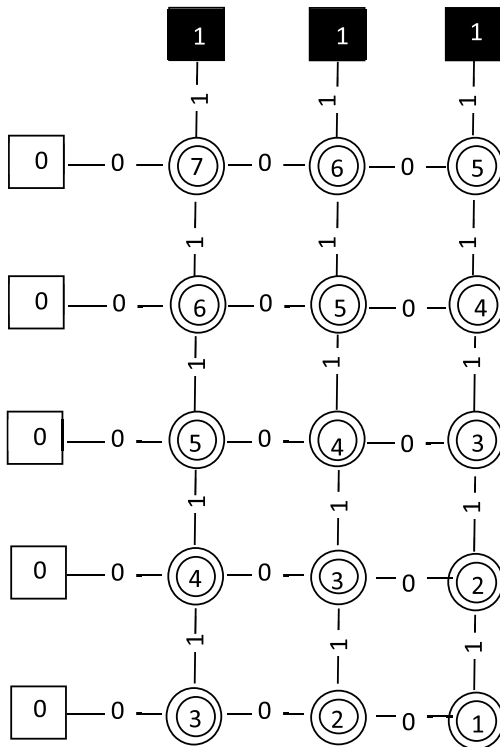
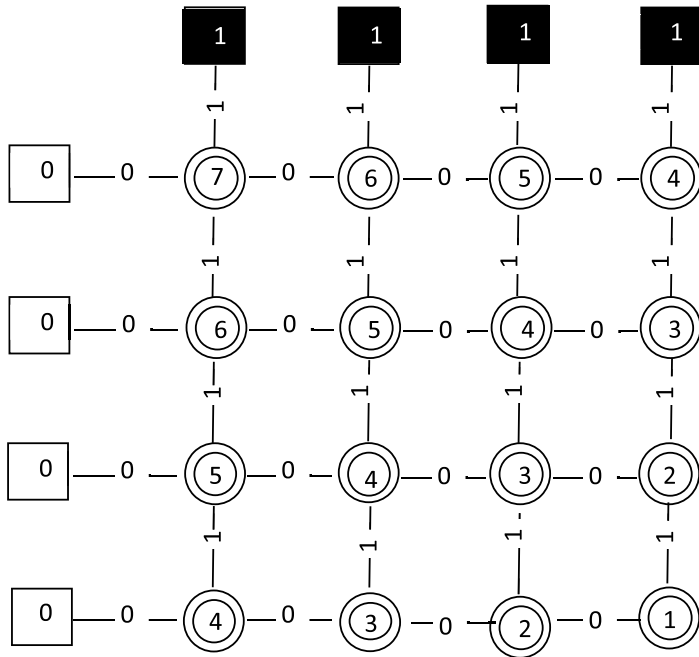


Fig. 4. The Signal Flow Graph representing the AR algorithm for computing the k_1 -out-of- (k_1+k_2) : G system up to $r = k_1+k_2 = 7$. Components reliabilities have a reverse order from 7 down to 1.





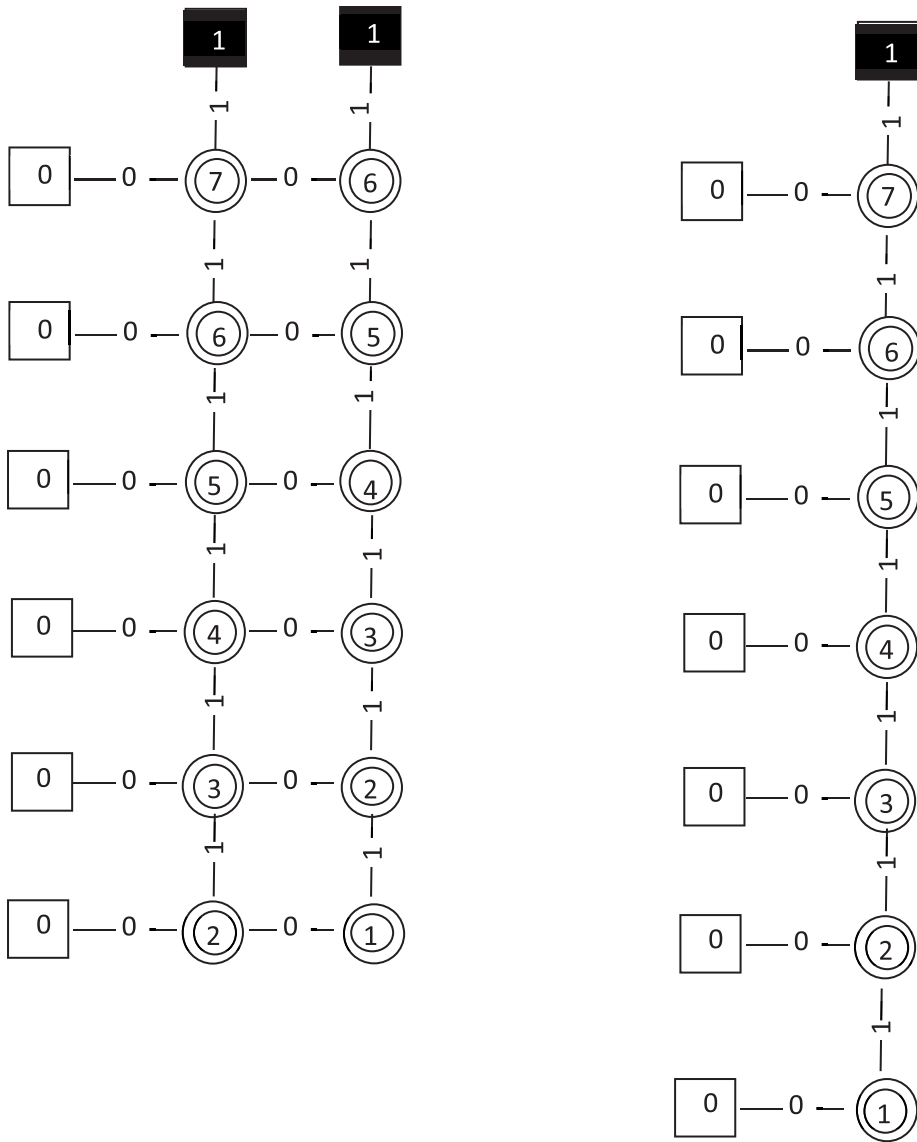


Fig. 5. The individual ROBDDs used in the computation of $R(k, 7, \mathbf{p})$, $1 \leq k \leq 7$. For simplicity, split leaf nodes of 0 and 1 are used.

Note that nodes in Figure 6 represent $E(k_1, k_1 + k_2, \mathbf{p}_{k_1 + k_2})$ values rather than $R(k_1, k_1 + k_2, \mathbf{p}_{k_1 + k_2})$ and hence are shown as single circles rather than double ones. This is a consequence of the fact that each of these figures is used simply to enumerate path transmittances for paths that emanate from a non-source node to another by replacing the node from which the paths originate by a source node of value 1. Note also that the subgraphs in Fig. 3 and Fig. 6 are the same, apart from a reordering of components. Figure 6 consequently produces Eq. (23), which is a special case of the general IMDP formula (17). Again, it is straightforward to generalize the current SFG derivation to general k and n .

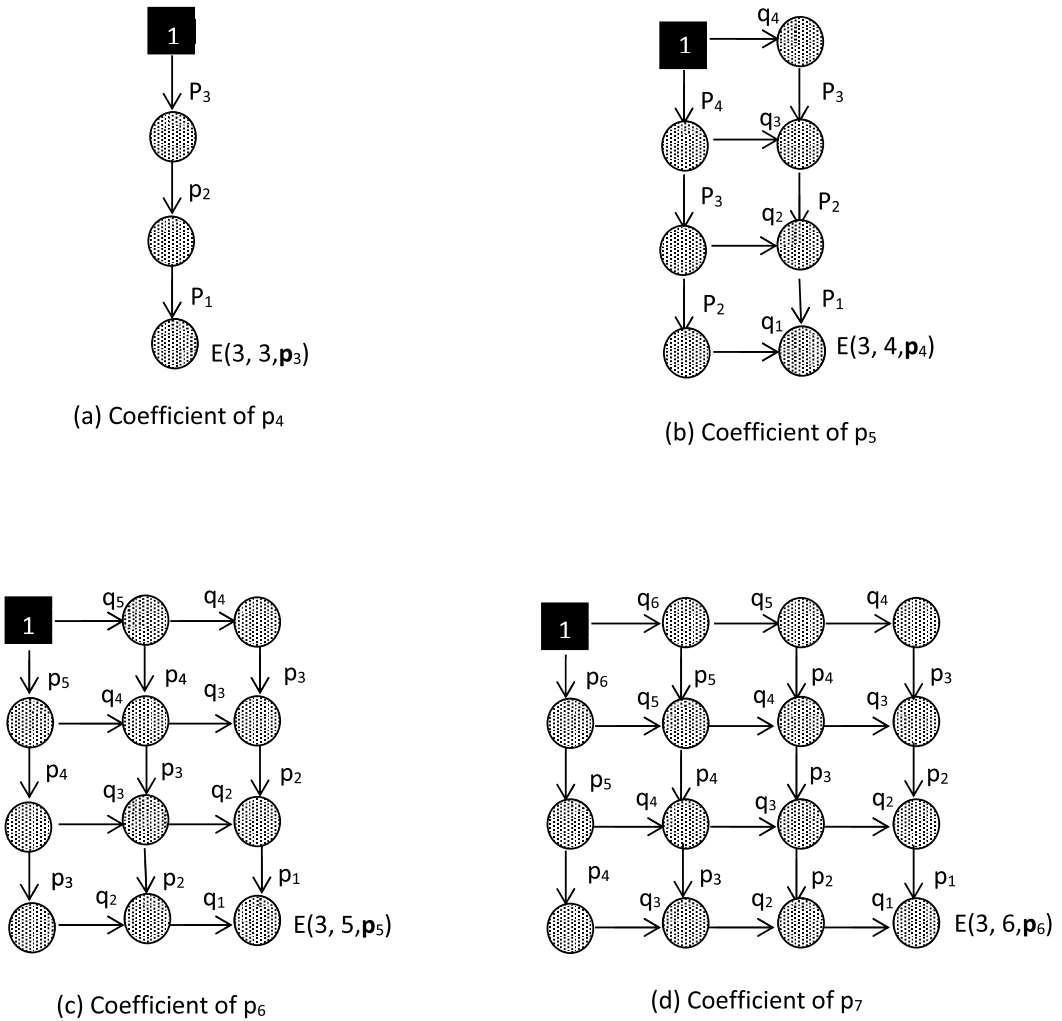


Fig. 6. A representation of the coefficients of p_l ($4 \leq l \leq 7$) in the expression of $R(4, 7, \mathbf{p})$ via the AR algorithm. These coefficients are consequently displayed as E values, that is, as single-node circles.

Figure 7 provides a pictorial summary of the relations between the IMDP, BH-2, and AR algorithms. The three algorithms are quite similar, and there is no significant preference of one of them over another. Figure 7 stresses the equivalence of the BH-2 and AR algorithms and asserts that the IMDP algorithm is a natural outcome of each of them.

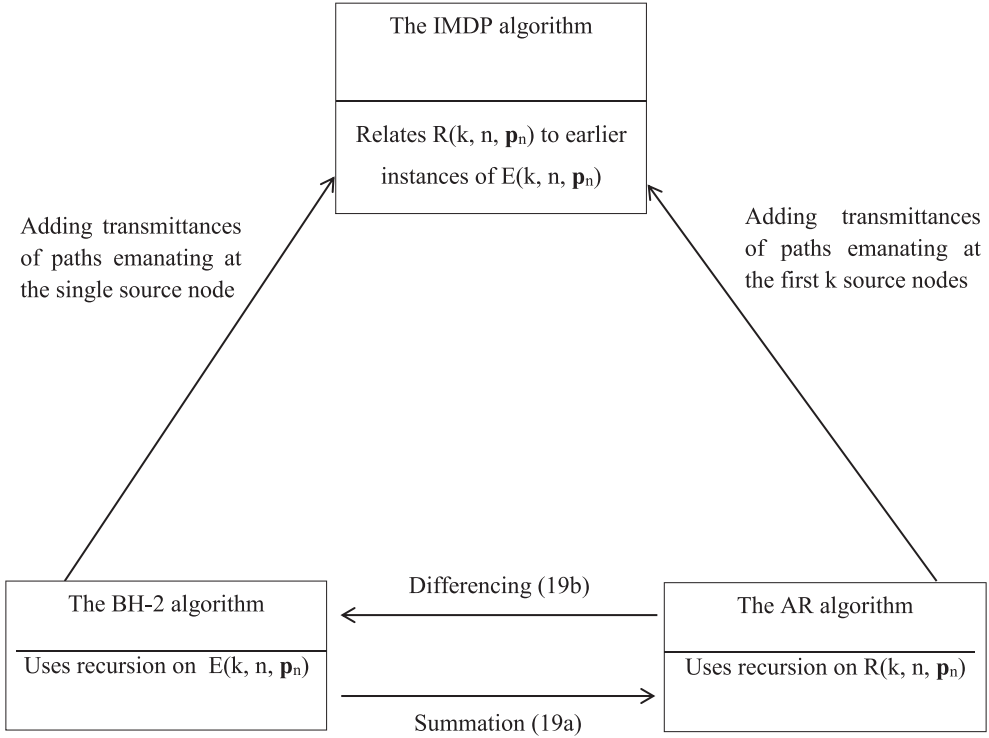


Fig. 7. Interrelations between the IMDP, BH-2, and AR algorithms.

THE IMDP ALGORITHM AS A COLLECTIVE SHELLABILITY ALGORITHM

The k -out-of- n : G system has success indicator variables given by the sum-of-products expression (Ruhdi, 1993)

$$S(k, n, \mathbf{X}) = \bigvee_{\mathbf{K}} \left(\bigwedge_{a \in \mathbf{K}} X_a \right), \quad (29)$$

where \mathbf{K} is a subset of $\{1, 2, \dots, n\}$ with a cardinality of $|\mathbf{K}| = k$. Equation (29) expresses S as an ORing of $\binom{n}{k}$ prime implicants of S , called minimal paths (tiesets) of the system. Similarly, the failure of the k -out-of- n : G system is given by the sum-of-products expression (Rushdi, 1993):

$$\bar{S}(k, n, \mathbf{X}) = \bigvee_{\mathbf{C}} \left(\bigwedge_{a \in \mathbf{C}} \bar{X}_a \right), \quad (30)$$

where \mathbf{C} is a subset of $\{1, 2, \dots, n\}$ with cardinality $|\mathbf{C}| = n - k + 1$. Equation (30) expresses \bar{S} as an ORing of $\binom{n}{n-k+1}$ prime implicants of \bar{S} , called minimal cutsets of the system. One has to work with either (29) (if $k \geq n - k + 1$) or (30) (if $k < n - k + 1$). Locks (1984) noted that disjointing terms in either (29) or (30) (to obtain a PRE) is possible without increasing the number of terms, which means, in modern language, that S in (29) and \bar{S} in (30) are both shellable. To achieve this, Locks

proposed that minimal products (minimal paths in (29) or minimal cuts in (30)) be arranged in an order such that each product differs from its predecessor in exactly one literal. However, he did not present a general strategy or an algorithm to achieve such an arrangement, and Rushdi (1993) demonstrated that this arrangement is a *sufficient* but *not a necessary* condition. Anyhow, Locks (1984) is credited with being the first to seek "shellability", even before this mathematical term was adopted for reliability applications. We now demonstrate that the IMDP algorithm is a general strategy to achieve shellability for either of the shellable expressions S in (29) or \bar{S} in (30). We consider the success function $S(4, 7, \mathbf{X})$ of a 4-out-of-7:G, which has $\binom{7}{4} = 35$ minimal paths of 4 literals each. These paths are arranged in Table 1 according to the IMDP formula (17) to groups of 1, 4, 10, and 20 terms, respectively. The first group has a single arbitrarily chosen product (here $X_1 X_2 X_3 X_4$). In the second group, an arbitrarily chosen literal of the 3 so far unused literals (here X_5) replaces a single literal of the initial product, that is, one at a time. Hence, $\binom{4}{3} = 4$ products are obtained, which constitute the second group. In the third group, an arbitrarily chosen literal of the 2 so far remaining literals (here X_6) replaces a single literal of the previous five literals in the predecessor products, thereby producing $\binom{5}{3} = 10$ products that constitute the third group. The fourth and final groups consist of terms in which the so far remaining literal (X_7) appears. The number of terms in this group is $\binom{6}{3} = 20$ terms. *The order of terms within each group is arbitrary.* Several comments on Table 1 are in order:

1. Table 1 demonstrates clearly that, with the IMDP arrangement, the shellability algorithm (Crama and Hammer, 2011) works perfectly.
2. The disjunction of terms in the last column up to rows 1, 5, 15, and 35 is, respectively, PRE expressions for the successes $S(4, 4, \mathbf{p4})$, $S(4, 5, \mathbf{p5})$, $S(4, 6, \mathbf{p6})$, and $S(4, 7, \mathbf{p7})$. Each of these successes is in the form suggested by the IMDP formula (17).
3. The entries in the column for $\bar{C}_1 \bar{C}_2 \dots \bar{C}_{i-1}$ in the rows $i = 2, 6, 16$, and 36 are symmetric functions representing the failures of the 1-out-of-4:F, 2-out-of-5:F, 3-out-of-6:F, and 4-out-of-7:F systems (which are the same as the 4-out-of-4:G, 4-out-of-5:G, 4-out-of-6:G, 4-out-of-7:G systems). Note that Table 1 includes an extra step (row 36), in which the shellability algorithm successfully produces the complementary function $\bar{S}(4, 7, \mathbf{X})$ which consists of $\binom{n}{n-k+1} = \binom{7}{4} = 35$ terms.
4. The intersection of the shadow $S(A_i)$ with any A_l preceding A_i is not \emptyset .
5. In each row, the entry in the last column $\bar{C}_1 \bar{C}_2 \dots \bar{C}_{i-1}$ equal to the logical product of the corresponding entries in the $\bar{C}_1 \bar{C}_2 \dots \bar{C}_{i-1}$ column and the C_i column. However, it is obtained instead by augmenting C_i by failures of the elements in the corresponding shadow set.
6. In each row, the entry for the product $\bar{C}_1 \bar{C}_2 \dots \bar{C}_{i-1}$ is obtained by multiplying the same entry in the preceding row (representing $\bar{C}_1 \bar{C}_2 \dots \bar{C}_{i-2}$) by the *complement* of the entry C_i in the preceding row (representing \bar{C}_{i-1}). The resulting product differs from the preceding one in a few literals that are highlighted in bold red. The multiplication needed for $\bar{C}_1 \bar{C}_2 \dots \bar{C}_{i-1}$ is quite involved, but it is much simplified by the rule of "Intelligent Multiplication" (Brown, 1990; Rushdi and Al-Yahya, 2001; Rushdi and BaRukab, 2014), namely,

$$(a \vee b) (a \vee c) = a \vee bc. \tag{31}$$

The result of multiplication in each row is minimized simply by deleting any term that subsumes another (Muroga, 1979; Rushdi & Al-Yahya, 2001). For example, in going from row 22 to row 23, we multiply the entry under $\bar{C}_1\bar{C}_2\dots\bar{C}_{i-1}$ by the complement of $X_2X_3X_5X_7$ which is $\bar{X}_2\bar{X}_3\bar{X}_5\bar{X}_7$. The first term in that entry becomes.

$$\begin{aligned} & \bar{X}_1\bar{X}_5\bar{X}_6\bar{X}_7 (\bar{X}_2\bar{X}_3\bar{X}_5\bar{X}_7) \\ &= \bar{X}_1\bar{X}_2\bar{X}_5\bar{X}_6\bar{X}_7\bar{X}_1\bar{X}_3\bar{X}_5\bar{X}_6\bar{X}_7\bar{X}_1\bar{X}_5\bar{X}_6\bar{X}_7\bar{X}_1\bar{X}_5\bar{X}_6\bar{X}_7 \\ &= \bar{X}_1\bar{X}_5\bar{X}_6\bar{X}_7, \end{aligned} \tag{32}$$

where each of the first three terms subsumes the last one and is absorbed in it. Therefore, the term $\bar{X}_1\bar{X}_5\bar{X}_6\bar{X}_7$ remains *intact* after multiplication. Similarly, all other terms remain intact, with the sole exception of the term $\bar{X}_1\bar{X}_4\bar{X}_6$ (corresponding to the complementary set for A_{22} or the shadow set), which becomes

$$\begin{aligned} & \bar{X}_1\bar{X}_4\bar{X}_6 (\bar{X}_2\bar{X}_3\bar{X}_5\bar{X}_7) \\ &= \bar{X}_1\bar{X}_2\bar{X}_4\bar{X}_6 \vee \bar{X}_1\bar{X}_3\bar{X}_4\bar{X}_6 \vee \bar{X}_1\bar{X}_4\bar{X}_5\bar{X}_6 \vee \bar{X}_1\bar{X}_4\bar{X}_6\bar{X}_7. \end{aligned} \tag{33}$$

Table 1. Demonstration that the IMDP algorithm achieves shellability for the shellable expression $S(4, 7, \mathbf{X})$.

i	C_i	Set A_i	Shadow $S(A_i)$	$\bar{C}_1\bar{C}_2\dots\bar{C}_{i-1}$	$\bar{C}_1\bar{C}_2\dots\bar{C}_{i-1}C_i$
1	$X_1X_2X_3X_4$	{1,2,3,4}	\emptyset	1	$(X_1X_2X_3)X_4$
2	$X_2X_3X_4X_5$	{2,3,4,5}	{1}	$\bar{X}_1 \vee \bar{X}_2 \vee \bar{X}_3 \vee \bar{X}_4$	$(\bar{X}_1X_2X_3X_4)X_5$
3	$X_1X_3X_4X_5$	{1,3,4,5}	{2}	$\bar{X}_1\bar{X}_5 \vee \bar{X}_2 \vee \bar{X}_3 \vee \bar{X}_4$	$(X_1\bar{X}_2X_3X_4)X_5$
4	$X_1X_2X_4X_5$	{1,2,4,5}	{3}	$\bar{X}_1\bar{X}_5 \vee \bar{X}_1\bar{X}_2 \vee \bar{X}_2\bar{X}_5 \vee \bar{X}_3 \vee \bar{X}_4$	$(X_1X_2\bar{X}_3X_4)X_5$
5	$X_1X_2X_3X_5$	{1,2,3,5}	{4}	$\bar{X}_1\bar{X}_5 \vee \bar{X}_1\bar{X}_2 \vee \bar{X}_2\bar{X}_5 \vee \bar{X}_1\bar{X}_3 \vee \bar{X}_2\bar{X}_3 \vee \bar{X}_3\bar{X}_5 \vee \bar{X}_4$	$(X_1X_2X_3\bar{X}_4)X_5$
6	$X_2X_3X_4X_6$	{2,3,4,6}	{1,5}	$\bar{X}_1\bar{X}_5 \vee \bar{X}_1\bar{X}_2 \vee \bar{X}_2\bar{X}_5 \vee \bar{X}_1\bar{X}_3 \vee \bar{X}_2\bar{X}_3 \vee \bar{X}_3\bar{X}_5 \vee \bar{X}_1\bar{X}_4 \vee \bar{X}_2\bar{X}_4 \vee \bar{X}_3\bar{X}_4 \vee \bar{X}_4\bar{X}_5$	$(\bar{X}_1X_2X_3X_4)X_6$
7	$X_1X_3X_4X_6$	{1,3,4,6}	{2,5}	$\bar{X}_1\bar{X}_5\bar{X}_6 \vee \bar{X}_1\bar{X}_2 \vee \bar{X}_2\bar{X}_5 \vee \bar{X}_1\bar{X}_3 \vee \bar{X}_2\bar{X}_3 \vee \bar{X}_3\bar{X}_5 \vee \bar{X}_1\bar{X}_4 \vee \bar{X}_2\bar{X}_4 \vee \bar{X}_3\bar{X}_4 \vee \bar{X}_4\bar{X}_5$	$(X_1\bar{X}_2X_3X_4)X_6$
8	$X_1X_2X_4X_6$	{1,2,4,6}	{3,5}	$\bar{X}_1\bar{X}_5\bar{X}_6 \vee \bar{X}_1\bar{X}_2 \vee \bar{X}_2\bar{X}_5\bar{X}_6 \vee \bar{X}_1\bar{X}_3 \vee \bar{X}_2\bar{X}_3 \vee \bar{X}_3\bar{X}_5 \vee \bar{X}_1\bar{X}_4 \vee \bar{X}_2\bar{X}_4 \vee \bar{X}_3\bar{X}_4 \vee \bar{X}_4\bar{X}_5$	$(X_1X_2\bar{X}_3X_4)X_6$
9	$X_1X_2X_3X_6$	{1,2,3,6}	{4,5}	$\bar{X}_1\bar{X}_5\bar{X}_6 \vee \bar{X}_1\bar{X}_2 \vee \bar{X}_2\bar{X}_5\bar{X}_6 \vee \bar{X}_1\bar{X}_3 \vee \bar{X}_2\bar{X}_3 \vee \bar{X}_3\bar{X}_5\bar{X}_6 \vee \bar{X}_1\bar{X}_4 \vee \bar{X}_2\bar{X}_4 \vee \bar{X}_3\bar{X}_4 \vee \bar{X}_4\bar{X}_5$	$(X_1X_2X_3\bar{X}_4)X_6$
10	$X_3X_4X_5X_6$	{3,4,5,6}	{1,2}	$\bar{X}_1\bar{X}_5\bar{X}_6 \vee \bar{X}_1\bar{X}_2 \vee \bar{X}_2\bar{X}_5\bar{X}_6 \vee \bar{X}_1\bar{X}_3 \vee \bar{X}_2\bar{X}_3 \vee \bar{X}_3\bar{X}_5\bar{X}_6 \vee \bar{X}_1\bar{X}_4 \vee \bar{X}_2\bar{X}_4 \vee \bar{X}_3\bar{X}_4 \vee \bar{X}_4\bar{X}_5\bar{X}_6$	$(\bar{X}_1\bar{X}_2X_3X_4)X_6$
11	$X_2X_4X_5X_6$	{2,4,5,6}	{1,3}	$\bar{X}_1\bar{X}_5\bar{X}_6 \vee \bar{X}_1\bar{X}_2\bar{X}_5 \vee \bar{X}_1\bar{X}_2\bar{X}_6 \vee \bar{X}_2\bar{X}_5\bar{X}_6 \vee \bar{X}_1\bar{X}_3 \vee \bar{X}_2\bar{X}_3 \vee \bar{X}_3\bar{X}_5\bar{X}_6 \vee \bar{X}_1\bar{X}_4 \vee \bar{X}_2\bar{X}_4 \vee \bar{X}_3\bar{X}_4 \vee \bar{X}_4\bar{X}_5\bar{X}_6$	$(\bar{X}_1X_2\bar{X}_3X_4)X_6$
12	$X_2X_3X_5X_6$	{2,3,5,6}	{1,4}	$\bar{X}_1\bar{X}_5\bar{X}_6 \vee \bar{X}_1\bar{X}_2\bar{X}_5 \vee \bar{X}_1\bar{X}_2\bar{X}_6 \vee \bar{X}_2\bar{X}_5\bar{X}_6 \vee \bar{X}_1\bar{X}_3\bar{X}_5 \vee \bar{X}_1\bar{X}_3\bar{X}_6 \vee \bar{X}_2\bar{X}_3 \vee \bar{X}_3\bar{X}_5\bar{X}_6 \vee \bar{X}_1\bar{X}_4 \vee \bar{X}_2\bar{X}_4 \vee \bar{X}_3\bar{X}_4 \vee \bar{X}_4\bar{X}_5\bar{X}_6$	$(\bar{X}_1X_2X_3\bar{X}_4)X_6$
13	$X_1X_4X_5X_6$	{1,4,5,6}	{2,3}	$\bar{X}_1\bar{X}_5\bar{X}_6 \vee \bar{X}_1\bar{X}_2\bar{X}_5 \vee \bar{X}_1\bar{X}_2\bar{X}_6 \vee \bar{X}_2\bar{X}_5\bar{X}_6 \vee \bar{X}_1\bar{X}_3\bar{X}_5 \vee \bar{X}_1\bar{X}_3\bar{X}_6 \vee \bar{X}_2\bar{X}_3 \vee \bar{X}_3\bar{X}_5\bar{X}_6 \vee \bar{X}_1\bar{X}_4\bar{X}_5 \vee \bar{X}_1\bar{X}_4\bar{X}_6 \vee \bar{X}_2\bar{X}_4 \vee \bar{X}_3\bar{X}_4 \vee \bar{X}_4\bar{X}_5\bar{X}_6$	$(X_1\bar{X}_2\bar{X}_3X_4)X_6$
14	$X_1X_3X_5X_6$	{1,3,5,6}	{2,4}	$\bar{X}_1\bar{X}_5\bar{X}_6 \vee \bar{X}_1\bar{X}_2\bar{X}_5 \vee \bar{X}_1\bar{X}_2\bar{X}_6 \vee \bar{X}_2\bar{X}_5\bar{X}_6 \vee \bar{X}_1\bar{X}_3\bar{X}_5 \vee \bar{X}_1\bar{X}_3\bar{X}_6 \vee \bar{X}_1\bar{X}_2\bar{X}_3 \vee \bar{X}_2\bar{X}_3\bar{X}_5 \vee \bar{X}_2\bar{X}_3\bar{X}_6 \vee \bar{X}_3\bar{X}_5\bar{X}_6 \vee \bar{X}_1\bar{X}_4\bar{X}_5 \vee \bar{X}_1\bar{X}_4\bar{X}_6 \vee \bar{X}_2\bar{X}_4 \vee \bar{X}_3\bar{X}_4 \vee \bar{X}_4\bar{X}_5\bar{X}_6$	$(X_1\bar{X}_2X_3\bar{X}_4)X_6$
15	$X_1X_2X_5X_6$	{1,2,5,6}	{3,4}	$\bar{X}_1\bar{X}_5\bar{X}_6 \vee \bar{X}_1\bar{X}_2\bar{X}_5 \vee \bar{X}_1\bar{X}_2\bar{X}_6 \vee \bar{X}_2\bar{X}_5\bar{X}_6 \vee \bar{X}_1\bar{X}_3\bar{X}_5 \vee \bar{X}_1\bar{X}_3\bar{X}_6 \vee \bar{X}_1\bar{X}_2\bar{X}_3 \vee \bar{X}_2\bar{X}_3\bar{X}_5 \vee \bar{X}_2\bar{X}_3\bar{X}_6 \vee \bar{X}_3\bar{X}_5\bar{X}_6 \vee \bar{X}_1\bar{X}_4\bar{X}_5 \vee \bar{X}_1\bar{X}_4\bar{X}_6 \vee \bar{X}_2\bar{X}_4 \vee \bar{X}_3\bar{X}_4 \vee \bar{X}_4\bar{X}_5\bar{X}_6$	$(X_1X_2\bar{X}_3\bar{X}_4)X_6$
16	$X_2X_3X_4X_7$	{2,3,4,7}	{1,5,6}	$\bar{X}_1\bar{X}_5\bar{X}_6 \vee \bar{X}_1\bar{X}_2\bar{X}_5 \vee \bar{X}_1\bar{X}_2\bar{X}_6 \vee \bar{X}_2\bar{X}_5\bar{X}_6 \vee \bar{X}_1\bar{X}_3\bar{X}_5 \vee \bar{X}_1\bar{X}_3\bar{X}_6 \vee \bar{X}_1\bar{X}_2\bar{X}_3 \vee \bar{X}_2\bar{X}_3\bar{X}_5 \vee \bar{X}_2\bar{X}_3\bar{X}_6 \vee \bar{X}_3\bar{X}_5\bar{X}_6 \vee \bar{X}_1\bar{X}_4\bar{X}_5 \vee \bar{X}_1\bar{X}_4\bar{X}_6 \vee \bar{X}_1\bar{X}_2\bar{X}_4 \vee \bar{X}_2\bar{X}_4\bar{X}_5 \vee \bar{X}_2\bar{X}_4\bar{X}_6 \vee \bar{X}_1\bar{X}_3\bar{X}_4 \vee \bar{X}_2\bar{X}_3\bar{X}_4 \vee \bar{X}_3\bar{X}_4\bar{X}_5 \vee \bar{X}_3\bar{X}_4\bar{X}_6 \vee \bar{X}_4\bar{X}_5\bar{X}_6$	$(\bar{X}_1X_2X_3X_4\bar{X}_5)X_7$
17	$X_1X_3X_4X_7$	{1,3,4,7}	{2,5,6}	$\bar{X}_1\bar{X}_5\bar{X}_6\bar{X}_7 \vee \bar{X}_1\bar{X}_2\bar{X}_5 \vee \bar{X}_1\bar{X}_2\bar{X}_6 \vee \bar{X}_2\bar{X}_5\bar{X}_6 \vee \bar{X}_1\bar{X}_3\bar{X}_5 \vee \bar{X}_1\bar{X}_3\bar{X}_6 \vee \bar{X}_1\bar{X}_2\bar{X}_3 \vee \bar{X}_2\bar{X}_3\bar{X}_5 \vee \bar{X}_2\bar{X}_3\bar{X}_6 \vee \bar{X}_3\bar{X}_5\bar{X}_6 \vee \bar{X}_1\bar{X}_4\bar{X}_5 \vee \bar{X}_1\bar{X}_4\bar{X}_6 \vee \bar{X}_1\bar{X}_2\bar{X}_4 \vee \bar{X}_2\bar{X}_4\bar{X}_5 \vee \bar{X}_2\bar{X}_4\bar{X}_6 \vee \bar{X}_1\bar{X}_3\bar{X}_4 \vee \bar{X}_2\bar{X}_3\bar{X}_4 \vee \bar{X}_3\bar{X}_4\bar{X}_5 \vee \bar{X}_3\bar{X}_4\bar{X}_6 \vee \bar{X}_4\bar{X}_5\bar{X}_6$	$(X_1\bar{X}_2X_3X_4\bar{X}_5)X_7$
18	$X_1X_2X_4X_7$	{1,2,4,7}	{3,5,6}	$\bar{X}_1\bar{X}_5\bar{X}_6\bar{X}_7 \vee \bar{X}_1\bar{X}_2\bar{X}_5 \vee \bar{X}_1\bar{X}_2\bar{X}_6 \vee \bar{X}_2\bar{X}_5\bar{X}_6\bar{X}_7 \vee \bar{X}_1\bar{X}_3\bar{X}_5 \vee \bar{X}_1\bar{X}_3\bar{X}_6 \vee \bar{X}_1\bar{X}_2\bar{X}_3 \vee \bar{X}_2\bar{X}_3\bar{X}_5 \vee \bar{X}_2\bar{X}_3\bar{X}_6 \vee \bar{X}_3\bar{X}_5\bar{X}_6 \vee \bar{X}_1\bar{X}_4\bar{X}_5 \vee \bar{X}_1\bar{X}_4\bar{X}_6 \vee \bar{X}_1\bar{X}_2\bar{X}_4 \vee \bar{X}_2\bar{X}_4\bar{X}_5 \vee \bar{X}_2\bar{X}_4\bar{X}_6 \vee \bar{X}_1\bar{X}_3\bar{X}_4 \vee \bar{X}_2\bar{X}_3\bar{X}_4 \vee \bar{X}_3\bar{X}_4\bar{X}_5 \vee \bar{X}_3\bar{X}_4\bar{X}_6 \vee \bar{X}_4\bar{X}_5\bar{X}_6$	$(X_1X_2\bar{X}_3X_4\bar{X}_5)X_7$

CONCLUSIONS AND FUTURE WORK

This paper reviews and exposes a few old mathematical results as well as introduces and demonstrates many new ones for the evaluation of k-out-of-n:G reliability. Pertinent algorithms such as the ROBDD algorithm, the AR algorithm, the BH-2 algorithm, the IMDP algorithm, and the shellability algorithm are all interrelated. Several examples are also given to demonstrate the important concept of Signal Flow Graph (SFG) representation for recursive relations and their boundary conditions. This representation provides useful insight, and it was previously used for constructing iterative versions out of recursive ones (such as the iterative AR algorithm (Rushdi, 1986; 1991)). It is utilized herein as an aid in deriving useful new relations and in proving equivalences among different algorithms.

Despite the existence of several earlier tutorial reviews on the topic of k-out-of-n system reliability, this paper is believed to fill in a gap in this topic by unifying, updating, and modernizing many mathematical concepts pertinent to this topic. Though the paper is basically intended to provide many novel contributions, it is also useful as an expanded tutorial exposition on the subject.

The k-out-of-n model adopted herein does not consider time explicitly and hence cannot address the more realistic situation of load sharing, in which every component could be operated, derated with a lower failure rate. However, this simple model sets the stage for comprehending and then constructing any useful Markov chain model (Koutras, 1996), that is, powerful enough to cover load sharing. Possible future investigation is to utilize the algorithms presented herein in a finite Markov chain imbedding approach (Fu, 1986; Chao & Fu, 1989; 1991; Fu & Koutras, 1994; Boutsikas & Koutras, 2000, Cui *et al.*, 2010). This investigation is highly promising since imbedding the k-out-of-n structure in a Markov chain requires certain *recurrence relations* with *initial conditions* (Koutras, 1996). These relations are essentially those used herein in the BH-2 and AR algorithms. In fact, Koutras (1996) points out that his algorithm is “*actually the dual counterpart of Barlow and Heidtmann (1984) algorithm.*”

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توحيد المفاهيم الرياضية والخوارزميات لمعولية النظم الوافرة جزئيا: منظور الطريقة المحسنة للمضروبات المتعامدة

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الخلاصة

إن نموذج النظام الوافر جزئيا (نظام ك من بين ن) هو أبرز نماذج معولية النظم المتسقة، وهو يتميز بتعدد الحالات الخاصة الهامة له، فضلا عن كثرة تعميماته وتمديداته. وبصفة خاصة تمثل معوليات ك من بين ن أساسا يستند إليه في التعبير عن معولية نظام متسق نوني الرتبة بدلالة التوقيع (طيف التدمير) لهذا النظام. ثمة خوارزمية مرموقة لحساب معولية نظام ك من بين ن تسمى الطريقة المحسنة للمضروبات المتعامدة (ط ح ض ع) (IMDP). لورقة البحث هذه أربعة أهداف، وهي بالتحديد: (أ) تقديم شرح تفصيلي مبتكر لخوارزمية ط ح ض ع، و (ب) إيضاح إمكانية اشتقاق خوارزمية ط ح ض ع من خوارزمية BH-2 التي تعد تحسينا لخوارزمية BH-1، حيث تستعمل هذه الأخيرة لتقدير احتمال حدوث ك من النجاحات خلال ن من محاولات برنولي، ومن ثم لحساب دالة الكتلة الاحتمالية (د ك ح) للتوزيع ذي الحدين المعمم، و (ج) بيان إمكانية اشتقاق خوارزمية ط ح ض ع من خوارزمية AR التي تعد خوارزمية مخطط القرار الثنائي المرتب المختزل (خ ق ث ر خ) (ROBDD) لتقدير معولية ك من بين ن وأيضا لحساب دالة التوزيع التراكمي (د و ر) للتوزيع ذي الحدين المعمم، و (د) إظهار أن خوارزمية ط ح ض ع هي خوارزمية لتحقيق التعامد (التنافي) على الإجمال لصيغة متوقعة لمجموع مضروبات تعبر عن نجاح ك من بين ن. تلعب ورقة البحث دورا توحيدا للعديد من المفاهيم والخوارزميات، وتحاول التأكيد على أوجه الشبه والارتباطات والتداخلات بينها، في الوقت الذي تشير فيه إلى أية فروق دقيقة بينها. إن العامل المشترك في شرح الخوارزميات المختلفة هو استعمال رسوم لسريان الإشارة (ر س ش) (SFG) تتسم بكونها ملمومة ومنظمة وغير حلقية. ولهذه الرسوم عديمة الحلقات يحتاج قانون الكسب فقط إلى تعداد بسيط للمسارات ثم حساب المناقلات لهذه المسارات.